MATH 101 REVIEW: DERIVATIVES AND APPLICATIONS

Math 100 main topics:

- Real numbers, functions, absolute values, inequalities
- Limits and rates of change: limits of sequences and functions, limit laws, continuity, Intermediate Value Theorem
- Derivatives: tangents and differentiability, higher derivatives, differentiation formulae (including chain rule), implicit differentiation, Mean Value Theorem and applications (monotonicity, concavity)
- Elementary functions: inverse functions and their derivatives, derivatives of trig and inverse trig functions, exponential and logarithmic functions and their derivatives, exponential growth and decay
- Applications: curve sketching, maximum and minimum problems, related rate problems, l'Hopital's rule
- Approximation: linearization (with error estimate), quadratic and higher approximations, Taylor polynomials

Review problems I: True or false? Explain why.

(a) If $f(x) \le g(x) \le h(x)$ on some open interval including a, and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

(a) A function cannot be odd and even at the same time.

(b) A function continuous on a closed interval $[a, b]$ attains its minimum and maximum values.

(c) A function continuous on an open interval (a, b) attains its minimum and maximum values.

(d) If a function $f(x)$ is differentiable and increasing on an open interval (a, b), then $f'(x)$ is nonnegative on (a, b). Does $f'(x)$ have to be strictly positive on (a, b)?

(e) If $f'(x)$ exists and is positive at all points x where $f(x)$ is defined, then $f(x)$ is increasing.

(f) If $f(x)$ has a local minimum at a, then $f'(a) = 0$.

(g) If $f(x)$ is concave up on an open interval containing a and if $f'(a) = 0$, then $f(x)$ has a local minimum at a.

(h) If $f(x)$ is defined and twice differentiable on an open interval containing a, and if $f(x)$ has a local minimum at a, then there is an open interval containing a on which $f(x)$ is concave up.

(i) If a function $f(x)$ is continuous on the closed interval $[1, 2]$, $f(1) = -1$, $f(2) = 4$, then $f(x) = 3$ for at least one x in $[1, 2]$.

(j) If a function $f(x)$ is differentiable on the open interval $(1, 2)$, $f(1) = -1$, $f(2) = 4$, then $f'(x) = 5$ for at least one x in $(1, 2)$.

(k) If a function $f(x)$ and its derivative $f'(x)$ are continuous on the closed interval $[0, 2]$, $f(0) = 0$, $f(1) = 4$, $f(2) = 2$, then $f'(x) = 0$ for some x in $(0, 2)$.

Review problems II: Calculation and evaluation problems

1. Evaluate the limits

$$(a)\ \lim_{x\to 2} \frac{x^2-4}{x^2-2x},\ (b)\ \lim_{x\to\infty} (\sqrt{x^2+x} - x).$$

2. Compute the derivative of $f(x) = \frac{1}{x^2}$ directly from the definition.

3. Calculate and simplify where possible the derivatives of the following fuctions.

$$(a)\ \frac{\sin 5x}{1+x^2},\ (b)\ \ln(x+e^x),\ (c)\ (1+x^2)\tan^{-1}(x).$$

4. Use l'Hospital's rule to evaluate $\displaystyle\lim_{x\to 0} \frac{\sin(2x) - 2xe^{-x^2}}{x^3}$.

5. Sketch the graphs of the functions below. Identify any critical points, singular points, local maxima and minima, inflection points, vertical and horizontal asymptotes.

$$(a)\ f(x) = \frac{x^2-4}{x^2-1},\ (b)\ g(x) = xe^{-x^2/2},\ (c)\ h(x) = (x^2-1)^{2/3}.$$

6. Check that $e^{x+x^2} > 1 + x$ for all $x > 0$.

7. Find all lines that are tangent to both of the curves $y = x^2$ and $y = -x^2 + 2x - 5$. (Draw a picture.)

8. Find the equation of the tangent line to the curve $x \sin(xy - y^2) = x^2 - 1$ at $(1, 1)$.