MATH 120 MIDTERM 1 - SOLUTIONS

1. *Evaluate the following limits if they exist; if they do not exist, explain why.*

   (a)

$$\lim_{x\to 2}\frac{x^2-3x+2}{x^2-5x+6}=\lim_{x\to 2}\frac{(x-1)(x-2)}{(x-2)(x-3)}$$
$$=\lim_{x\to 2}\frac{x-1}{x-3}=\frac{2-1}{2-3}=-1.$$

   (b)

$$\lim_{x\to 1}\frac{\sqrt{x^2+1}-\sqrt{x+1}}{x-1}$$
$$=\lim_{x\to 1}\frac{(\sqrt{x^2+1}-\sqrt{x+1})(\sqrt{x^2+1}+\sqrt{x+1})}{(x-1)(\sqrt{x^2+1}+\sqrt{x+1})}$$
$$=\lim_{x\to 1}\frac{(x^2+1)-(x+1)}{(x-1)(\sqrt{x^2+1}+\sqrt{x+1})}$$
$$=\lim_{x\to 1}\frac{x^2-x}{(x-1)(\sqrt{x^2+1}+\sqrt{x+1})}$$
$$=\lim_{x\to 1}\frac{x}{\sqrt{x^2+1}+\sqrt{x+1}}=\frac{1}{\sqrt{2}+\sqrt{2}}=\frac{1}{2\sqrt{2}}.$$

2. *The function $f(x)$ is defined on the interval $[0,2]$ and is between $4-x$ and x^2+2 for all x in this interval. Does it have to be continuous at $x=1$? Explain why or why not.*

   Let $g(x)=4-x$ and $h(x)=x^2+2$. Then $g(1)=h(1)=3$, hence $f(1)=3$ as well. Also, g and h are continuous everywhere, and in particular $\lim_{x\to 1} g(x)=\lim_{x\to 1} h(x)=3$. By the squeeze theorem, $\lim_{x\to 1} f(x)=3=f(1)$, so that $f(x)$ is continuous at $x=1$.

3. *We know that* $f(x)$ *is a differentiable function such that* $f(0) = -3$ *and* $f'(0) = 2$. *What is* $\frac{d}{dx} f^2(2f(x)+6)|_{x=0}$*?*

We have

$$\frac{d}{dx} f^2(2f(x)+6) = 2f(2f(x)+6) \cdot \frac{d}{dx} f(2f(x)+6)$$
$$= 2f(2f(x)+6) \cdot f'(2f(x)+6) \cdot 2f'(x).$$

Plugging in $x = 0$, we get

$$2f(0) \cdot f'(0) \cdot 2f'(0) = 2 \cdot (-3) \cdot 2 \cdot 2 \cdot 2 = -48.$$

4. *Find the equation of the line tangent to the curve* $y = \sqrt{x^2 - 4x - 1}$ *at the point* $(-1, 2)$.

We have

$$y' = \frac{1}{2\sqrt{x^2-4x-1}} \cdot (2x-4) = \frac{x-2}{\sqrt{x^2-4x-1}}.$$

At $x = -1$, $y' = \frac{-1-2}{\sqrt{1+4-1}} = \frac{-3}{\sqrt{4}} = -\frac{3}{2}$. Therefore the equation of the tangent line is

$$y - 2 = -\frac{3}{2}(x+1), \text{ or } y = -\frac{3}{2}x + \frac{1}{2}.$$

5. *Prove that the equation* $x^7 - 3x - 1 = 0$ *has at least one solution in the interval* $-1 \le x \le 1$.

Let $f(x) = x^7 - 3x - 1$, then $f(-1) = -1 + 3 - 1 = 1 > 0$ and $f(1) = 1 - 3 - 1 = -3 < 0$. Since f is continuous, by the Intermediate Value Theorem there is at least one x in $[-1, 1]$ where $f(x) = 0$.