6 marks 1. (a) Let $S \subseteq \mathbb{R}$ be a set. Define precisely what it means for S to be well ordered.

Solution: S is well ordered if every non-empty subset of S has a least element.

(b) Let $g : A \to B$ be a function. Define precisely what it means for g to be surjective.

Solution: g is surjective when for every $y \in Y$ there exists $x \in X$ such that $g(x) = y$.

(c) Let $h : A \to B$ be a function. Define precisely what it means for h to be injective.

Solution: h is injective when for all $a_1, a_2 \in A$, $h(a_1) = h(a_2) \Rightarrow a_1 = a_2$. Equivalently $a_1 \neq a_2 \Rightarrow h(a_1) \neq h(a_2)$.

6 marks 2. Let $m, n \in \mathbb{Z}$. Prove that if $m \equiv n \pmod 3$ then $m^3 \equiv n^3 \pmod 9$.

Solution: We will use a direct proof.

Proof. Let $m \equiv n \pmod 3$, then $m - n = 3k$ for some $k \in \mathbb{Z}$. We can write $n = 3a + r$, where $a \in \mathbb{Z}$ and $r = 0, 1,$ or 2. Then $m = n + 3k = 3a + r + 3k = 3b + r$, where $b = a + k \in \mathbb{Z}$. Therefore

$$\begin{aligned} m^3 - n^3 &= (3a + r)^3 - (3b + r)^3 \\ &= (27a^3 + 27a^2 r + 9ar^2 + r^3) - (27b^3 + 27b^2 r + 9br^2 + r^3) \\ &= 9(3a^3 + 3a^2 r + ar^2 - 3b^3 - 3b^2 r - br^2) \end{aligned}$$

This is divisible by 9, since $3a^3 + 3a^2 r + ar^2 - 3b^3 - 3b^2 r - br^2 \in \mathbb{Z}$.

□

This could also be proved by cases (with $m \equiv n \equiv 0$, 1, and 2 (mod 3)). The calculation would be similar.

10 marks

3. Let $X = \{a + b\sqrt{2} : \ a, b \in \mathbb{Z}\}$.

(a) (2 marks) Prove that if $x, y \in X$, then $xy \in X$.

Solution:

Proof. Let $x, y \in X$, then $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$ for some $a, b, c, d \in \mathbb{Z}$. Then

$$\begin{aligned} xy &= (a + b\sqrt{2})(c + d\sqrt{2}) = ac + bc\sqrt{2} + ad\sqrt{2} + 2bd \\ &= (ac + 2bd) + (bc + ad)\sqrt{2}. \end{aligned}$$

This is in X, since $ac + 2bd$ and $bc + ad$ are in $\mathbb{Z}$. □

(b) (4 marks) Prove by induction that if $x \in X$, then $x^n \in X$ for every $n \in \mathbb{N}$.

Solution:

Proof. Let $x \in X$.

- The base case: let $n = 1$, then $x^1 = x$, so $x \in X$ by assumption.
- The inductive step: assume that $x \in X$ and $x^k \in X$, and let $n = k + 1$. Then $x^n = x^{k+1} = x^k x$. Applying part (a) with $y = x^k$, we get that $x^{k+1} \in X$.
- By induction, $x^n \in X$ for all $n \in \mathbb{N}$.

□

(c) (4 marks) Disprove the following statement:

If $x, y \in X$ and $y \neq 0$, then $\frac{x}{y} \in X$.

(Hint: You may use that $\sqrt{2}$ is irrational.)

Solution: We disprove this by counterexample.

- Let $x = 1 = 1 + 0\sqrt{2}$, $y = 2 = 2 + 0\sqrt{2}$. Then $x, y \in X$ and $y \neq 0$.
- We have $\frac{y}{x} = \frac{1}{2}$.
- Suppose that $1/2 \in X$. Then $1/2 = a + b\sqrt{2}$ for some $a, b \in \mathbb{Z}$, so that $1 = 2a + 2b\sqrt{2}$, $1 - 2a = 2b\sqrt{2}$.
- If $b = 0$, we get $1 - 2a = 0$, $a = 1/2$. But this is a contradiction, since $1/2$ is not an integer.
- If $b \neq 0$, we get $\sqrt{2} = \frac{1 - 2a}{2b}$. Since $1 - 2a$ and $2b$ are integers, $\sqrt{2}$ is rational. But this is again a contradiction.
- This proves that $1/2$ is not in X. So, we have our counterexample.

6 marks 4. A sequence $\{a_n\}$ is defined recursively by $a_1 = 0$, $a_2 = 1/3$, and $a_n = \frac{1}{3}(1 + a_{n-1} + a_{n-2}^2)$ for $n > 2$. Prove that $a_{n+1} > a_n$ for all $n \geq 1$.

Solution: We use the Strong Principle of Mathematical Induction.

Proof.

- If $n = 1$, $a_2 = \frac{1}{3} > 0 = a_1$.
- If $n = 2$, we have $a_2 = 1/3$ and $a_3 = \frac{1}{3}(1 + \frac{1}{3} + 0) = \frac{4}{9} > \frac{1}{3}$.
- Let $k \geq 2$, and assume that $a_{j+1} > a_j$ for $j = 1, 2, \ldots, k$. Then also $a_{j+1}^2 > a_j^2$ for $j = 1, 2, \ldots, k$. Now let's compare a_{k+2} and a_{k+1}:

$$a_{k+2} = \frac{1}{3}(1 + a_{k+1} + a_k^2) > \frac{1}{3}(1 + a_k + a_k^2) > \frac{1}{3}(1 + a_k + a_{k-1}^2) = a_{k+1}$$

(We used that $a_{k+1} > a_k$ and $a_k^2 > a_{k-1}^2$.)

- By induction, $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

□

6 marks 5. Let $f : \mathbb{R} - \{-3\} \to \mathbb{R} - \{2\}$ be the function defined by

$$f(x) = \frac{2x}{x+3}.$$

Prove that f is bijective.

Solution:

- We first prove that f is one to one. Suppose that $f(x) = f(y)$ for some $x, y \in \mathbb{R} - \{3\}$, then $\frac{2x}{x+3} = \frac{2y}{y+3}$,

$$2x(y+3) = 2y(x+3), \quad 2xy + 6x = 2yx + 6y,$$

so that $6x = 6y$, $y = x$. So f is one to one as required.

- We now prove that f is onto: for every $y \neq 2$, there is an $x \in \mathbb{R} - \{3\}$ such that $f(x) = y$, that is, $\frac{2x}{x+3} = y$. We solve this for x:

$$2x = y(x+3) = yx + 3y, \quad x(2-y) = 3y$$

If $y \neq 2$, there is a solution $x = \frac{3y}{2-y}$. We check that $x \neq -3$: if $\frac{3y}{2-y} = -3$, then $3y = -3(2-y) = 6 + 3y$, $0 = 6$, a contradiction. And finally,

$$f\left(\frac{3y}{2-y}\right) = \frac{2\frac{3y}{2-y}}{\frac{3y}{2-y} + 3} = \frac{6y}{3y + 6 - 3y} = \frac{6y}{6} = y$$