*This midterm has **5 questions** on **6 pages** Duration: 50 minutes*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (Last, First): ______________________________

Student Number: ______________________________

Signature: ______________________________

Question:	1	2	3	4	5	Total
Points:	12	8	12	16	12	60
Score:						

12 marks

1. (a) (4 marks) Give an example of a set A such that $|A| = 3$ and $\mathcal{P}(A) \cap A \neq \emptyset$.

Solution: $A = \{1, 2, \{1\}\}$.

(b) (4 marks) State the converse and the contrapositive of the following statement:

If $A \subseteq B$ then $A \cap C \subseteq B \cap C$.

Solution:

- Converse: If $A \cap C \subseteq B \cap C$ then $A \subseteq B$.
- Contrapositive: If $A \cap C \not\subseteq B \cap C$ then $A \not\subseteq B$.

(c) (4 marks) State the negation of the following statement:

$$\forall x \in A\ \exists y \in B\ \forall z \in C \quad x < z \Rightarrow y < x$$

Solution:

$$\exists x \in A\ \forall y \in B\ \exists z \in C \quad x < z\ \wedge\ y \geq x$$

8 marks 2. Let P, Q, R be statements. Show that

$$[\sim(P \Rightarrow Q) \vee \sim(P \Rightarrow R)] \quad \equiv \quad \sim(P \Rightarrow (Q \wedge R))\,.$$

Solution:

$$\begin{aligned}
[\sim(P \Rightarrow Q) \vee \sim(P \Rightarrow R)] &\equiv \sim[(P \Rightarrow Q) \wedge (P \Rightarrow R)] \\
&\equiv \sim[(\sim P \vee Q) \wedge (\sim P \vee R)] \\
&\equiv \sim[\sim P \vee (Q \wedge R)] \\
&\equiv \sim[P \Rightarrow (Q \vee R)]
\end{aligned}$$

A proof by truth table would also work.

12 marks 3. Prove the following statement: for all $n \in \mathbb{Z}$, the number $n^3 - 4n$ is divisible by 3.

Solution:

Proof by cases: $n \equiv 0, 1, 2 \pmod 3$.

- If $n \equiv 0 \pmod 3$, then $n = 3k$ for some $k \in \mathbb{Z}$. Then

$$n^3 - 4n = (3k)^3 - 4(3k) = 27k^3 - 12k = 3(9k^3 - 4k).$$

  Since $9k^3 - k \in \mathbb{Z}$, $n^3 - 4n$ is divisible by 3.

- If $n \equiv 1 \pmod 3$, then $n = 3k + 1$ for some $k \in \mathbb{Z}$. Then

$$n^3 - 4n = (3k+1)^3 - 4(3k+1) = 27k^3 + 27k^2 + 9k + 1 - 12k - 4$$
$$= 27k^3 + 27k^2 - 3k - 3 = 3(9k^3 + 9k^2 - k - 1).$$

  Since $9k^3 + 9k^2 - k - 1 \in \mathbb{Z}$, $n^3 - 4n$ is divisible by 3.

- If $n \equiv 2 \pmod 3$, then $n = 3k + 2$ for some $k \in \mathbb{Z}$. Then

$$n^3 - 4n = (3k+2)^3 - 4(3k+2) = 27k^3 + 54k^2 + 36k + 8 - 12k - 8$$
$$= 27k^3 + 54k^2 + 24k = 3(9k^3 + 18k^2 + 8k).$$

  Since $9k^3 + 18k^2 + 8k \in \mathbb{Z}$, $n^3 - 4n$ is divisible by 3.

It is also possible to expand $n^3 - 4n = n(n^2 - 4) = n(n-2)(n+2)$ and then work by cases. In that solution, the calculations are a little bit easier.

16 marks

4. Determine whether each of the following statement is True or False. Justify your answer.

(a) (4 marks) Let $A = \{\emptyset\}$. For every set B, $A \times B = \emptyset$.

Solution: False. Take $B = \{1\}$. Then $A \times B = \{(\emptyset, 1)\}$.

(b) (4 marks) $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ s.t. $yz = x$.

Solution: False. Take $x = 1, y = 0$. Then $\forall z \in \mathbb{R}$, $yz = 0 \neq 1 = x$.

(c) (4 marks) Let $n \in \mathbb{Z}$ and $a, b \in \mathbb{N}$. If $a|n$ and $b|n$ then $(ab)|n$.

Solution: False. Take $n = 4, a = 4, b = 2$. We have $4|4, 2|4, (4 \times 2) \nmid 4$.

(d) (4 marks) $\forall x, y \in \mathbb{R}$, if $x \neq y$ then $x^2 + y^2 > 0$.

Solution: True. If $x \neq y$, then x, y cannot be both 0. Say $x \neq 0$, then $x^2 > 0$. Since $y^2 \geq 0$. Then $x^2 + y^2 > 0$.

12 marks 5. Let A and B be sets. Show that $(A \cup B) - B = A$ if and only if $A \cap B = \emptyset$.

Solution: Assume $(A \cup B) - B = A$. If $x \in A \cap B$, then $x \in A, x \in B$. Since $x \in A = (A \cup B) - B$, $x \notin B$. So no such x can exist. Thus, $A \cap B = \emptyset$.

Assume $A \cap B = \emptyset$.

If $x \in (A \cup B) - B$, then $x \in A \cup B$, so $x \in A$ or $x \in B$. Then, $x \notin B$ implies $x \in A$. Hence, $(A \cup B) - B \subseteq A$.

If $x \in A$, then $x \notin B$ because $A \cap B = \emptyset$. $x \in A$ also implies $x \in A \cup B$. So, $x \in (A \cup B) - B$. Hence $A \subseteq (A \cup B) - B$. We now conclude $(A \cup B) - B = A$.