24 marks 1. (a) Write the converse, contrapositive and negation of the following statement:

For every integer n, if n is odd then n^2 is odd.

statement: $\forall n \in \mathbb{Z} \quad n \text{ odd} \Rightarrow n^2 \text{ odd}$

converse: $\forall n \in \mathbb{Z} \quad n^2 \text{ odd} \Rightarrow n \text{ odd}$

Contrapositive: $\forall n \in \mathbb{Z} \quad n^2 \text{ even} \Rightarrow n \text{ even}$

negation: $\exists n \in \mathbb{Z}$ s.t. $n \text{ odd} \wedge n^2 \text{ even}$

(b) Let A be a nonempty set and let S be a collection of some of the subsets of A. What properties are required for S to be called a partition of A?

$A \neq \emptyset \qquad S \subseteq \mathcal{P}(A)$

We say S is a partition of A if:

(1) $\forall B \in S \quad B \neq \emptyset$

(2) $\forall B, C \in S \quad B = C \vee B \cap C = \emptyset$

(3) $\bigcup_{B \in S} B = A$

(c) Let A_n be the interval $[-1+\frac{1}{n}\ ,\ 2+\frac{1}{n^2})$ for $n \in \mathbb{N}$. Find $\bigcap_{n\in\mathbb{N}} A_n$ and $\bigcup_{n\in\mathbb{N}} A_n$.

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} [-1+\tfrac{1}{n}, 2+\tfrac{1}{n^2}) = [0,3) \cap [-\tfrac{1}{2}, 2+\tfrac{1}{4}) \cap [-\tfrac{2}{3}, 2+\tfrac{1}{9}) \cap \cdots$$

$$= [0,2]$$

notice that $-1+\frac{1}{n} \xrightarrow[n\to\infty]{} -1$ and $2+\frac{1}{n^2} \xrightarrow[n\to\infty]{} 2$

So from the left side, these intervals are getting larger towards -1, while -1 does not belong to any and from the right side, these intervals are getting smaller towards 2 and two (2) belongs to all of them.

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} [-1+\tfrac{1}{n}, 2+\tfrac{1}{n^2}) = [0,3) \cup [-\tfrac{1}{2}, 2+\tfrac{1}{4}) \cup [-\tfrac{2}{3}, 2+\tfrac{1}{9}) \cup \cdots$$

$$= (-1, 3)$$

(d) Is the following statement true or false? Justify your answer.

For every real number a, there exist two natural numbers b and c such that if $b < a < c$ then $b^2 < a^2 < c^2$.

$\forall a \in \mathbb{R} \quad \exists b, c \in \mathbb{N}$ s.t. $b < a < c \Rightarrow b^2 < a^2 < c^2$

This statement is true because if $a \leq 1$ we can choose $b = 1$ and $c = 2$ then since $b < a < c$ is false, the implication is true.

If $a > 1$ then choose $b = 1$ and c an arbitrary natraal bigger than a. Then $1 < a < c$ and $1 < a^2 < c^2$ are both true.

8 marks 2. Let $n \in \mathbb{Z}$. Prove that $n^2 + 5n + 7$ is odd.

Proof with cases:

Case 1. If n is ~~odd~~ then

$\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$

then $n^2 + 5n + 7 = (2k+1)^2 + 5(2k+1) + 7$

$= 4k^2 + 4k + 1 + 10k + 5 + 7$

$= 4k^2 + 14k + 13$

$= 2(2k^2 + 2k + 6) + 1$ is odd.

Case 2. If n is ~~od~~ even then

$\exists k \in \mathbb{Z}$ s.t. $n = 2k$

then $n^2 + 5n + 7 = 4k^2 + 10k + 7$

$= 2(2k^2 + 5k + 3) + 1$ is odd.

∎

8 marks 3. Prove the following statement and disprove its converse.

"For all three integers a, b, c; if a does not divide bc then a does not divide b and a does not divide c."

$\forall a, b, c \in \mathbb{Z} \quad a \nmid bc \Rightarrow (a \nmid b \wedge a \nmid c)$

Proof by contrapositive:

We prove that $\forall a, b, c \in \mathbb{Z} \quad (a \mid b \vee a \mid c) \Rightarrow a \mid bc$

If $a \mid b$ or $a \mid c$ then $b = ak$ for some integer k or $c = al$ for some integer l

So $bc = a(kc)$ or $bc = a(bl)$ in either case $a \mid bc$.

Disproof of the converse: $\forall a, b, c \in \mathbb{Z} \quad (a \nmid b \wedge a \nmid c) \Rightarrow a \nmid$

counterexample: $a = 6 \quad b = 2 \quad c = 3$

then $6 \nmid 2 \wedge 6 \nmid 3$ but $6 \mid 2 \times 3$.

12 marks 4. Let n and m be two integers. Prove that if $n \equiv 1 \pmod 6$ and $m \equiv 3 \pmod 8$ then $n^2 + 5m \equiv 0 \pmod 8$.

Proof (Direct proof)

We want to show that:

$$\forall n, m \in \mathbb{Z};\ \left(n \overset{6}{\equiv} 1 \ \wedge\ m \overset{8}{\equiv} 3\right) \Longrightarrow n^2 + 5m \overset{8}{\equiv} 0$$

Assume $n, m \in \mathbb{Z}$ such that $n \overset{6}{\equiv} 1$ and $m \overset{8}{\equiv} 3$

then

$$6 \mid (n-1) \ \wedge\ 8 \mid (m-3)$$

$$\Rightarrow n-1 = 6k \ \wedge\ m-3 = 8l \quad \text{with some } k, l \in \mathbb{Z}$$

$$\Rightarrow n = 6k+1 \ \wedge\ m = 8l+3$$

$$\Rightarrow n^2 + 5m = 36k^2 + 12k + 1 + 40l + 15$$

$$\overset{8}{\equiv} 4k^2 + 4k$$

since $8 \mid (32k^2 + 8k + 40l + 16)$

So $n^2 + 5m \overset{8}{\equiv} 4k(k+1)$

Also $2 \mid k(k+1)$ for all $k \in \mathbb{Z}$
since k and $k+1$ are two successive integers

therefore $8 \mid n^2 + 5m$. $\blacksquare$

12 marks 5. Remember that $\mathcal{P}(A)$ denotes the power set of A.

(a) Let A, B be sets. Prove that if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Direct proof: Let $S \in \mathcal{P}(A)$ be arbitrary then $S \subseteq A$. Since $A \subseteq B$ we have that $S \subseteq B$ So $S \in \mathcal{P}(B)$.

■

(b) Give an example that shows $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

Let $A = \{1\}$ $B = \{2\}$

then $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\}$

$= \{\emptyset, \{1\}, \{2\}\}$

but $A \cup B = \{1, 2\}$

and $\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

12 marks 6. (a) For statements P, Q and R, show that the two compound statements $(P \Rightarrow Q) \Rightarrow R$ and $((\sim P) \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent.

$$
\begin{aligned}
(P \Rightarrow Q) \Rightarrow R &\equiv \sim (P \Rightarrow Q) \vee R \\
&\equiv \sim (\sim P \vee Q) \vee R \\
&\equiv (P \wedge \sim Q) \vee R \\
&\equiv (P \vee R) \wedge (\sim Q \vee R) \\
&\equiv (\sim P \Rightarrow R) \wedge (Q \Rightarrow R)
\end{aligned}
$$

This can be proved using a truth table too.

(b) Let A, B, C be sets. Prove that $A - (B \cap C) = (A - B) \cup (A - C)$.

$$
\begin{aligned}
A - (B \cap C) = A \cap (\overline{B \cap C}) &= A \cap (\bar{B} \cup \bar{C}) \\
&= (A \cap \bar{B}) \cup (A \cap \bar{C}) \\
&= (A - B) \cup (A - C)
\end{aligned}
$$

12 marks 7. Prove that $\sqrt{10}$ is an irrational number.

Proof by Contradiction:

Assume $\sqrt{10} \in \mathbb{Q}$ then

$\exists\, m, n \in \mathbb{N}$ s.t. $\sqrt{10} = \frac{m}{n} \wedge \gcd(m,n) = 1$

greatest common divisor.

$\sqrt{10} = \frac{m}{n} \implies \left(\frac{m}{n}\right)^2 = 10 \implies m^2 = 10n^2 = 2(5n^2)$

$\implies 2 \mid m^2$

$\implies 2 \mid m$ (we need a lemma here which can be proven easily)

$\forall m \in \mathbb{Z} \quad 2 \mid m \iff 2 \mid m^2$

$\implies m = 2k$ with some integer k

$\implies m^2 = 4k^2$

$m^2 = 4k^2 = 10n^2 \implies 5n^2 = 2k^2$

$\implies 2 \mid 5n^2 \underset{*}{\implies} 2 \mid n^2 \implies 2 \mid n$

$2 \mid m \wedge 2 \mid n \implies \gcd(m,n) \neq 1$ contradicting our assumption

*(To prove $2 \mid 5n^2 \implies 2 \mid n$, assume $2 \nmid n$

then $n = 2l + 1$ then $5n^2 = 5(4l^2 + 4l + 1) = 20l^2 + 20l + 5$
$= 2(10l^2 + 10l + 2) + 1$ is odd.)

12 marks 8. Prove or disprove: If a, b and c be three positive integers such that $a^2 + b^2 + c^2 = 8m + 3$ for some integer m, then all a, b and c are odd.

$\forall\, a, b, c \in \mathbb{N} \quad a^2 + b^2 + c^2 \overset{8}{\equiv} 3 \implies$ a and b and c are odd.

Proof. Based on symmetry, there are four cases either we have three even, or two even one odd, or two odd one even or three odd.

Case 1 (three even) $a = 2k, b = 2l, c = 2h$

Then $a^2 + b^2 + c^2 = 4k^2 + 4l^2 + 4h^2$
$= 4(k^2 + l^2 + h^2)$

is even but $8m + 3 = 2(4m+1) + 1$ is odd.

Case 2 (two even one odd) $a = 2k \quad b = 2l \quad c = 2h+1$

$a^2 + b^2 + c^2 = 4k^2 + 4l^2 + 4h^2 + 4h + 1$
$= 4(k^2 + l^2 + h^2 + h) + 1$

$a^2 + b^2 + c^2 \overset{4}{\equiv} 1$

but $8m + 3 \overset{4}{\equiv} 3$.

Case 3 (two odd one even) $a = 2k+1 \quad b = 2l+1 \quad c = 2h$

$a^2 + b^2 + c^2 = 4k^2 + 4k + 1 + 4l^2 + 4l + 1 + 4h^2$

$= 4k^2 + 4l^2 + 4h^2 + 4k + 4l + 2$

is even

but $8m + 3$ is odd.

So only case 4 is possible. ∎