*This midterm has **6 questions** on **7 pages**, for a total of 100 points.*

Duration: 80 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	20	20	10	20	20	10	100
Score:							

20 marks

1. (a) (10 marks) For $A = \{-2, -1, 0, 1, 2\}$ and $B = \{-3, -2, \ldots, 2, 3\}$ list the elements of the set

$$S = \left\{(a, b) \in A \times B \ : \ (a - b)^2 = 4\right\}.$$

Solution: We have $(a, b) \in S$ if and only if either $a - b = 2$ or $a - b = -2$. Therefore

$$S = \left\{(-2, 0), (-1, -3), (-1, 1), (0, 2), (0, -2), (1, 3), (1, -1), (2, 0)\right\}.$$

(b) (10 marks) For the sets A, B defined in part (a), give an example of a set X such that $A \cap B \subset X \subset A \cup B$.

Solution: We have $A \cap B = A$ and $A \cup B = B$, so we could take either $C = \{-2, -1, 0, 1, 2, 3\}$ or $C = \{-3, -2, -1, 0, 1, 2\}$.

20 marks 2. (a) (10 marks) Let P and Q be statements. Show that

$$[(P \vee Q) \wedge \sim(P \wedge Q)] \quad \equiv \sim(P \Leftrightarrow Q)\,.$$

Solution:

P	Q	$\sim(P \Leftrightarrow Q)$	$[(P \vee Q) \wedge \sim(P \wedge Q)]$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

(b) (10 marks) Prove or disprove the following statement. Let A, B, C be sets, then

$$A - (B - C) \subseteq (A - B) \cup (A \cap C)\,.$$

Solution: True. Let $x \in A - (B - C)$ so that $x \in A$ and $x \notin B - C$. By De Morgan this means that $x \in A$ and ($x \notin B$ or $x \in C$). So we either have that ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \in C$) so $x \in (A - B) \cup (A \cap C)$.

10 marks 3. Let $m, n \in \mathbb{Z}$. Prove that if $m \equiv n \pmod 3$ then $m^3 \equiv n^3 \pmod 9$.

Solution: We will use a direct proof.

Proof. Let $m \equiv n \pmod 3$, then $m - n = 3k$ for some $k \in \mathbb{Z}$. We can write $n = 3a + r$, where $a \in \mathbb{Z}$ and $r = 0, 1$, or 2. Then $m = n + 3k = 3a + r + 3k = 3b + r$, where $b = a + k \in \mathbb{Z}$. Therefore

$$\begin{aligned} m^3 - n^3 &= (3a+r)^3 - (3b+r)^3 \\ &= (27a^3 + 27a^2r + 9ar^2 + r^3) - (27b^3 + 27b^2r + 9br^2 + r^3) \\ &= 9(3a^3 + 3a^2r + ar^2 - 3b^3 - 3b^2r - br^2) \end{aligned}$$

This is divisible by 9, since $3a^3 + 3a^2r + ar^2 - 3b^3 - 3b^2r - br^2 \in \mathbb{Z}$.

□

This could also be proved by cases (with $m \equiv n \equiv 0$, 1, and 2 (mod 3)). The calculation would be similar.

20 marks 4. (a) (10 marks) Prove the following statement: for all $n \in \mathbb{Z}$, the number $n^2 + n - 1$ is odd.

Solution: We prove this by cases.

- Suppose that $n \in \mathbb{Z}$ is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$, so

$$n^2 + n - 1 = (2k+1)^2 + (2k+1) - 1 = 4k^2 + 4k + 1 + 2k + 1 - 1$$
$$= 4k^2 + 6k + 1 = 2(2k^2 + 3k) + 1$$

  Since $2k^2 + 3k$ is integer, $n^2 + n - 1$ is odd.

- Suppose now that $n \in \mathbb{Z}$ is even, then $n = 2k$ for some $k \in \mathbb{Z}$, so

$$n^2 + n - 1 = (2k)^2 + (2k) - 1 = 4k^2 + 2k - 1 = 2(2k^2 + k - 1) + 1$$

  Since $2k^2 + k - 1$ is integer, $n^2 + n - 1$ is odd.

The conclusion is true since it is true in both cases.

(b) (10 marks) Let a, b, c be integers. Prove that if $a^2 \nmid bc$, then either $a \nmid b$ or $a \nmid c$.

Solution: We will prove the contrapositive, which is

If $a|b$ and $a|c$, then $a^2|bc$.

Assume that $a|b$ and $a|c$, then $b = ax$ and $c = ay$ for some $x, y \in \mathbb{Z}$. Then $bc = (ax)(ay) = a^2xy$ and $xy \in \mathbb{Z}$, so $a^2|bc$.

20 marks 5. (a) (10 marks) Prove that every real number x satisfies $x^2 + 4 > |2x - 1|$.

Solution: Let $x \in \mathbb{R}$. We have two cases.

- If $2x - 1 \geq 0$, then $|2x - 1| = 2x - 1$, so we have to prove $x^2 + 4 > 2x - 1$, or equivalently $x^2 - 2x + 5 > 0$. But $x^2 - 2x + 5 = (x - 1)^2 + 4 \geq 4 > 0$.
- If $2x - 1 < 0$, then $|2x - 1| = -2x + 1$, so we have to prove $x^2 + 4 > -2x + 1$, or equivalently $x^2 + 2x + 3 > 0$. But $x^2 + 2x + 3 = (x + 1)^2 + 2 \geq 2 > 0$.

(b) (10 marks) Prove the following statement: $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $y \leq x - 1 \Rightarrow y^2 - x^2 \geq 4$.

Solution: Let $x = -2$. Then $y \leq x - 1$ implies $y \leq -3$, so that $x + y \leq -2 - 3 = -5$. Also, $y \leq x - 1$ implies $y - x \leq -1$. Therefore $y^2 - x^2 = (y + x)(y - x) \geq (-5)(-1) = 5$.

(No additional points for trying to find the optimal x.)

10 marks 6. Prove that the number $\sqrt{2}+2\sqrt{3}$ is irrational. (In this question, you may use that $\sqrt{2}$ and $\sqrt{3}$ are irrational.)

Solution: Proof by contradiction: Suppose that $\sqrt{2}+2\sqrt{3}$ is rational, then $\sqrt{2}+2\sqrt{3}=\frac{m}{n}$ for some $m,n\in\mathbb{Z}$, $n\neq 0$. Then $\sqrt{2}=\frac{m}{n}-2\sqrt{3}$. Squaring this, we get

$$2=\frac{m^2}{n^2}-4\frac{m}{n}\sqrt{3}+12,$$

$$4\frac{m}{n}\sqrt{3}=\frac{m^2}{n^2}+10.$$

- If $m\neq 0$, we get
$$\sqrt{3}=\frac{\frac{m^2}{n^2}+10}{4m/n}=\frac{m^2+10n^2}{4mn}.$$
Since m^2+10n^2 and $4mn$ are both integer, $\sqrt{3}$ is rational, which is a contradiction.
- If $m=0$, then $\sqrt{2}+2\sqrt{3}=0$, so that $\sqrt{2}=-2\sqrt{3}$. Squaring this, we get $2=12$, a contradiction again.