This midterm has ***7 questions*** *on* ***8 pages****, for a total of 100 points.*

Duration: 80 minutes

- Read all the questions carefully before starting to work.
- You should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (including all middle names): ____________________________

Student-No: ____________________________

Signature: ____________________________

Question:	1	2	3	4	5	6	7	Total
Points:	14	16	16	8	12	18	16	100
Score:								

14 marks 1. (a) (8 marks) Let P be the statement

If 3 is even, then 6 is even or divisible by 5.

Write out the negation, converse and contrapositive of P. Determine the truth value of all four statements (P, its negation, converse and contrapositive). Explain your answers.

Solution:

- P is true because 3 is not even. ("P is true because 6 is even" is also a correct answer.)
- Negation: 3 is even and 6 is neither even nor divisible by 4. False, because 3 is not even (alternatively, because 6 is even).
- Converse: If 6 is even or divisible by 5, then 3 is even. False, because 6 is even but 3 is not even.
- Contrapositive: If 6 is neither even nor divisible by 5, then 3 is not even. True because 3 is not even (alternatively, because 6 is even).

(b) (6 marks) Is the statement

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall z \in \mathbb{Z}, \text{ if } z > y \text{ then } z > x.$$

true or false? Explain why. Write the negation of it.

Solution: The statement is true: given $x \in \mathbb{R}$, let $y = x$ (for example), then the conclusion holds. The negation is:

$$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{Z}, z > y \text{ and } z < x.$$

16 marks

2. (a) (4 marks) For $n \in \mathbb{N}$, let $A_n = \{x \in \mathbb{R} : 1 - \frac{1}{n} \leq x \leq n^2\}$. Find $\bigcap_{n \in \mathbb{N}} A_n$ and $\bigcup_{n \in \mathbb{N}} A_n$.

Solution:

$$\bigcap_{n \in \mathbb{N}} A_n = \{x \in \mathbb{R} : 1 \leq x \leq 1\} = \{1\}, \quad \bigcup_{n \in \mathbb{N}} A_n = [0, \infty)$$

(b) (4 marks) For the sets A_n from part (a), find $\bigcap_{n \in I} A_n$ and $\bigcup_{n \in I} A_n$, if $I = \{2, 3, 4\}$.

Solution: $A_2 = [1/2, 4]$, $A_3 = [2/3, 9]$, $A_4 = [3/4, 16]$. Therefore

$$\bigcap_{n \in I} A_n = [3/4, 4], \quad \bigcup_{n \in I} A_n = [1/2, 16]$$

(c) (4 marks) Is there a non-empty set $J \subset \mathbb{N}$ such that $\bigcap_{n \in J} A_n = \bigcup_{n \in J} A_n$? If yes, find such J (one example will suffice); if no, explain why.

Solution: Yes, J could be any one-element set. For example, $J = \{1\}$, then $\bigcap_{n \in J} A_n = \bigcup_{n \in J} A_n = A_1$.

(d) (4 marks) Give an example of a set B such that $B \cap \mathcal{P}(B) \neq \emptyset$. (Recall that $\mathcal{P}(B)$ is the power set of B.)

Solution: We can take $B = \{1, \{1\}\}$ then $\{1\} \in B \cap \mathcal{P}(B)$.

16 marks 3. (a) (8 marks) Prove that for all $n \in \mathbb{N}$, the number $5n + 3$ is odd if and only if $3n - 2$ is even.

Solution: Proof by cases.

- Let $n \in \mathbb{N}$, and assume that n is odd, so that $n = 2k + 1$ for some integer k. Then $5n+3 = 5(2k+1)+3 = 10k+8 = 2(5k+4)$ and $5k+4$ is integer, so $5n + 3$ is even. Also, $3n - 2 = 3(2k + 1) - 2 = 6k + 1 = 2(3k) + 1$ is odd. Therefore the statements

  "$5n + 3$ is odd" and "$3n - 2$ is even"

  are both false, so the biconditional is true in this case.

- Assume now that n is even, so that $n = 2k$ for some integer k. Then $5n + 3 = 10k + 3 = 2(5k + 1) + 1$ is odd, since $5k + 1$ is integer. Also, $3n - 2 = 6k - 2 = 2(3k - 1)$ is even, since $3k - 1 \in \mathbb{Z}$. Therefore in this case both statements are true, so that the biconditional is again true.

(b) (8 marks) Let a, b, c be integers such that $a \neq 0$. Prove that if a divides $b + c$ and a does not divide b, then a does not divide c.

Solution: Proof by contradiction: suppose that $a|b + c$, $a \not| b$, and $a|c$. Then $b+c = ka$ and $c = \ell a$ for some integers k, ℓ. But then $b = (b+c)-c = ka-\ell a = (k - \ell)a$, so that a divides b, which contradicts our assumptions.

8 marks 4. Prove the following statement: for all $n \in \mathbb{Z}$, if n^2 is not divisible by 4 then $n^2 \equiv 1 \pmod 4$.

Solution: We prove this by cases.

- Suppose that $n \in \mathbb{Z}$ is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$, so

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1 \equiv 1 \pmod 4$$

  In this case, n^2 is not divisible by 4 and $n^2 \equiv 1 \pmod 4$, so the implication is true.

- Suppose now that $n \in \mathbb{Z}$ is even, then $n = 2k$ for some $k \in \mathbb{Z}$, so $n^2 = (2k)^2 = 4k^2$ is divisible by 4 since $k \in \mathbb{Z}$. Hence the implication is again true.

12 marks

5. Are the following statements true or false? Prove your answers.

(a) (6 marks) Let A, B, C be sets. If $A \cap B$, $B \cap C$, and $A \cap C$ are all non-empty, then $A \cap B \cap C$ is non-empty.

Solution: False: let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{1, 5\}$. Then $A \cap B = \{3\}$, $B \cap C = \{5\}$ and $A \cap C = \{1\}$ are all non-empty, but $A \cap B \cap C = \emptyset$.

(b) (6 marks) If A and B are sets such that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

Solution: True. We will prove the contrapositive, which is

If $A \nsubseteq B$, then $\mathcal{P}(A) \nsubseteq \mathcal{P}(B)$.

Suppose that $A \nsubseteq B$, then there is an element $x \in A$ such that $x \notin B$. Then $\{x\} \in \mathcal{P}(A)$, but $\{x\} \notin \mathcal{P}(B)$. Therefore $\mathcal{P}(A) \nsubseteq \mathcal{P}(B)$, as claimed.

18 marks

6. (a) (12 marks) Consider the statements

   (1) For every $x \in \mathbb{R}$, there exists a $y \in \mathbb{R}$ such that if $x+1 > y$ then $x > y$.

   (2) There exists $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, if $x+1 > y$ then $x > y$.

   For each statement, decide whether it is true or false. Prove your answer.

   Solution:

   - (1) is true. Given $x \in \mathbb{R}$, we can take $y = x - 1$, then $x + 1 > x - 1 = y$ and $x > x - 1 = y$, so the implication is true. (Any other y such that $y < x$ or $y \geq x + 1$ would also work.)
   - (2) is false. It says that there is an $x \in \mathbb{R}$ such that the implication "if $x + 1 > y$ then $x > y$" is true for <u>all</u> $y \in \mathbb{R}$. But this is false because given any x, if $y = x + .5$, then $x + 1 > y$ but $x < y$.

   (b) (6 marks) Prove that the number $\frac{\sqrt{2}+1}{4}$ is irrational. (You can use that $\sqrt{2}$ is irrational.)

   Solution: We will prove this by contradiction. Suppose that $\frac{\sqrt{2}+1}{4}$ is rational, then $\frac{\sqrt{2}+1}{4} = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. Then $\sqrt{2} + 1 = \frac{4m}{n}$, so that $\sqrt{2} = \frac{4m}{n} - 1 = \frac{4m - n}{n}$, so that $\sqrt{2}$ is rational. But we know this to be false. Therefore the original statement is true.

16 marks 7. (a) (8 marks) Let P, Q, R be statements. Prove that the statements

$$P \Rightarrow \Big((\sim Q) \vee (Q \Rightarrow R)\Big) \quad \text{and} \quad \sim \Big((P \wedge Q) \wedge (\sim R)\Big)$$

are logically equivalent.

Solution:

$$\begin{aligned}\sim \Big[P \Rightarrow \Big((\sim Q) \vee (Q \Rightarrow R)\Big)\Big] &\equiv P \wedge \sim \Big((\sim Q) \vee (Q \Rightarrow R)\Big) \\ &\equiv P \wedge \Big(Q \wedge \sim (Q \Rightarrow R)\Big) \\ &\equiv P \wedge \Big(Q \wedge (Q \wedge \sim R)\Big) \\ &\equiv P \wedge Q \wedge (\sim R)\end{aligned}$$

Negating both statements, we get the equivalence we need.
(This could also be done using a truth table as below.)

(b) (8 marks) Prove that for any two statements P, Q, the following is a tautology:

$$\Big(((\sim P) \Rightarrow Q) \wedge (\sim Q)\Big) \Rightarrow P$$

Solution: We will use a truth table. Let R denote the statement

$$((\sim P) \Rightarrow Q) \wedge (\sim Q)$$

Then

P	Q	$\sim P$	$(\sim P) \Rightarrow Q$	$\sim Q$	R	$R \Rightarrow P$
T	T	F	T	F	F	T
T	F	F	T	T	T	T
F	T	T	T	F	F	T
F	F	T	F	T	F	T