This final exam has ***8 questions*** *on* ***10 pages****, for a total of 90 marks.*

Duration: 2 hours 30 minutes

Full Name: Not Real

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Question:	1	2	3	4	5	6	7	8	Total
Points:	8	24	10	10	10	6	12	10	90
Score:									

8 marks 1. (a) Negate the following statement:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. if } |x-a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

Solution:

$$\exists \epsilon > 0 \text{ s.t. } \forall \delta > 0, (|x-a| < \delta) \text{ and } (|f(x) - L| \geq \epsilon)$$

(b) Let A and B be non-empty sets. What is the definition of a relation from A to B?

Solution: A relation R from A to B is a subset $R \subseteq A \times B$.

(c) Give the definition of a set being denumerable.

Solution: A set A is countable if it is finite or denumerable (ie in bijection with the set of natural numbers).

(d) State the principle of mathematical induction.

Solution: Let $P(n)$ be a statement for all $n \in \mathbb{N}$. If

- $P(1)$ is true, and
- $P(k) \Rightarrow P(k+1)$ is true for all $k \in \mathbb{N}$,

Then $P(n)$ is true for all $n \in \mathbb{N}$.

(e) Carefully define what it means for a graph to be a tree.

Solution: A graph is a tree when it is connected and does not contain any cycles.

(f) State the hand-shaking theorem.

Solution: Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \delta(v) = 2|E|$$

24 marks 2. Determine whether the following statements are true or false. Circle "TRUE" or "FALSE" in the appropriate box and then justify your answer (just circling "TRUE" or "FALSE" is not sufficient).

In parts (a),(b),(c) we use $\mathbb{I}$ to denote the set of irrational numbers.

(a) $\exists x \in \mathbb{I}$ s.t. $\exists q \in \mathbb{Q}$ s.t. $x \cdot q \in \mathbb{I}$

TRUE | FALSE

Solution: This is true. Let $x = \sqrt{2}$ and $q = 1$. Then $xq = \sqrt{2} \in \mathbb{I}$.

(b) $\exists x \in \mathbb{I}$ s.t. $\forall q \in \mathbb{Q}$, $x \cdot q \in \mathbb{I}$

TRUE | FALSE

Solution: This is false. The negation of the statement is

$$\forall x \in \mathbb{I}, \exists q \in \mathbb{Q} \text{ s.t. } x \cdot q \notin \mathbb{I}$$

Let x be any irrational number and set $q = 0 \in \mathbb{Q}$. Then $xq = 0 \in \mathbb{Q}$. Hence $xq \notin \mathbb{I}$. Since the negation is true, the original is false.

(c) $\forall x \in \mathbb{I}$, $\exists q \in \mathbb{Q}$ s.t. $x + q \in \mathbb{Q}$

TRUE | FALSE

Solution: This is false. It suffices to show that the sum of an irrational number and a rational number is always irrational. We prove this by contradiction. Assume $x + q = \frac{a}{b}$. Then $x = \frac{a}{b} - q$. Since the difference of rational numbers is rational, this implies that x is rational which gives a contradiction.

(d) Let A, B be non-empty sets and $f : A \to B$ and $g : B \to A$ be functions. If $g \circ f = i_A$ (the identity function), then f is injective.

TRUE | FALSE

Solution: This is true. Let $x, z \in A$ and assume $f(x) = f(z)$. Since g is a function we know that $g(f(x)) = g(f(z))$. Now since $g \circ f$ is the identity function it follows that $x = z$. Hence f is injective.

(e) Let R be a relation on $\mathbb{R}$ defined by aRb iff $|a-b| < 4$. Then R is an equivalence relation on $\mathbb{R}$.

TRUE | FALSE

Solution: False. Transitivity does not hold: take $a = 1, b = 2, c = 5$. So $aRb, bRc, a \not R c$.

(f) For any integers a, b, if $ab \equiv 0 \pmod 6$, then either $6|a$ or $6|b$.

TRUE | FALSE

Solution: False. Take $a = 2, b = 3$.

(g) Let $G = (V, E)$ be a simple graph with n vertices. Then $|E| \leq \frac{n(n-1)}{2}$.

TRUE | FALSE

Solution: True. Since the graph is simple each vertex has degree at most $n-1$. The handshaking theorem then says

$$2|E| = \sum_{v \in V} \delta(v) \leq \sum_{v \in V} (n-1) = n(n-1)$$

The result follows.

(h) Let $G = (V, E)$ be a simple planar graph with n vertices and e edges. If every face of G is surrounded by at least 4 edges then $e \leq 2n - 4$.

TRUE | FALSE

Solution: True. Euler's formula says $n + f = e + 2$. Since every face needs at least 4 edges, and each edge belongs to 2 faces, $4f \leq 2e$. Thus

$$\begin{aligned} e &= n + f - 2 \leq n + \frac{e}{2} - 2 \\ 2e &\leq 2n + e - 4 \end{aligned}$$

The result follows.

10 marks 3. (a) Let $f : A \to B$ be a function and let $C \subseteq A$. Prove that $f(A) - f(C) \subseteq f(A - C)$.

Solution: Let f be a function and $C \subseteq A$. Now let $y \in f(A) - f(C)$. Since $y \in f(A)$ it follows that there is some $x \in A$ so that $f(x) = y$. Now either $x \in C$ or $x \notin C$.

- If $x \in C$ then $f(x) \in f(C)$ (by definition). But this would imply that $y = f(x) \notin f(A) - f(C)$. Hence we must have that $x \notin C$.

But now, $x \in A$ and $x \notin C$. Thus $x \in A - C$, and so $y = f(x) \in f(A - C)$ as required.

(b) Construct an example that shows that $f(A - C) \nsubseteq f(A) - f(C)$.

Solution: Let $A = \{1, 2\}, B = \{3, 4\}$ and $C = \{2\}$. Let $f(1) = f(2) = 3$. Then $f(A) = f(C) = \{3\}$ and $f(A-C) = f(\{1\}) = \{3\}$. Hence $f(A-C) = \{3\}$ but $f(A) - f(C) = \varnothing$.

10 marks 4. Use induction to answer this question.

Prove that for all natural numbers n

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n}\right).$$

Hint — be careful with both sides of the equation.

Solution: We prove the result using induction.

- Base case — when $n=1$ we have $\frac{1}{1+1}=\frac{1}{2}=1-\frac{1}{2}$. Hence the statement is true when $n=1$.
- Inductive step — assume the statement is true when $n=k$:

$$\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right).$$

We must show that

$$\frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}=\left(1-\frac{1}{2}\right)+\cdots+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right).$$

So start by adding $\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right)$ to both sides. The RHS becomes

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right)$$

as required, and the LHS becomes

$$\begin{aligned}\frac{1}{k+1}+\cdots+\frac{1}{2k}+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right)&=\left(\frac{1}{k+2}+\cdots+\frac{1}{2k+1}\right)+\left(\frac{1}{k+1}-\frac{1}{2k+2}\right)\\&=\frac{1}{k+2}+\cdots\frac{1}{2k+1}+\frac{1}{2k+2}\end{aligned}$$

as required. Thus $P(k+1)$ is true.

By mathematical induction the statement is true for all $n\in\mathbb{N}$.

10 marks 5. (a) Let $g : [0,\infty) \to [0,1)$ be a function which is defined by $g(x) = \frac{x^2}{1+x^2}$. Is g surjective?

Solution: For any $c \in [0,1)$, we have $c/(1-c) \geq 0$ hence

$$x = \sqrt{c/(1-c)} \in [0,\infty).$$

Now, for this x, we have

$$\begin{aligned} g(x) &= \frac{x^2}{1+x^2} \\ &= \frac{c/(1-c)}{1+c/(1-c)} \\ &= \frac{c}{(1-c)+c} = c, \end{aligned}$$

and therefore g is surjective.

(b) Let $f : \mathbb{R} \to \mathbb{R} \times [0,1)$ be a function which is defined by $f(x) = \left(x^2 + x, \frac{x^2}{1+x^2}\right)$. Is f injective?

Solution: If $f(a) = f(b)$, then

$$a^2 + a = b^2 + b \qquad \text{and} \qquad a^2/(1+a^2) = b^2/(1+b^2).$$

From the first equation we have

$$\begin{aligned} a^2 - b^2 + (a-b) &= 0 \\ (a-b)(a+b) + (a-b) &= 0 \\ (a-b)(a+b-1) &= 0 \end{aligned}$$

Hence either $a = b$ or $a = 1 - b$.
The second equation gives us

$$\begin{aligned} a^2(1+b^2) &= b^2(a^2+1) \\ a^2 + a^2b^2 &= b^2a^2 + b^2 \\ a^2 - b^2 &= 0 \\ (a-b)(a+b) &= 0 \end{aligned} \qquad \text{and so}$$

Thus either $a = b$ or $a = -b$.
Thus we must have

- $a = b$ and $a = b$ and so $a = b$.
- $a = b$ and $a = -b$ and so $a = b = 0$.
- $a = 1 - b$ and $a = b$ and so $a = b = 1/2$.
- $a = 1 - b$ and $a = -b$ — which is impossible

So we must have $a = b$ and so f is injective.

6 marks 6. (a) Give an example of a set that has no proper subsets.

> **Solution:** Let A be the empty set. The only subset of A is the empty set itself. Hence A has no proper subset.

(b) Give an example of a set B that has a proper subset C so that $|B| = |C|$. Prove your answer (you must construct a bijection).

> **Solution:** Let $B = \mathbb{N}$ and let $C = \{2n \mid n \in \mathbb{N}\}$. Then $C \subsetneq B$ and the function $f : B \to C$ defined by $f(n) = 2n$ is a bijection.
>
> - Injective — let $a, b \in \mathbb{N}$ so that $f(a) = f(b)$. Hence $2a = 2b$ and thus $a = b$. So f is injective.
> - Surjective — let $c \in C$. By the definition of the set C, $c = 2n$ for some $n \in \mathbb{N}$. Hence $f(n) = 2n = c$. So f is surjective.

12 marks 7. Let $n \in \mathbb{N}$.

(a) Prove that if $3|n$ then $3|n^3$.

Solution: Assume that $3|n$ and so $n = 3k$ for some $k \in \mathbb{Z}$. Then $n^3 = 27k^3 = 3(9k^3)$. Hence $3|n^3$.

(b) Prove that if $3|n^3$ then $3|n$.

Solution: We prove the contrapositive. Assume that $3 \nmid n$. Then $n = 3k+1$ or $n = 3k+2$ for some $k \in \mathbb{Z}$. Then

- when $n = 3k+1$, $n^3 = 27k^3 + 27k^2 + 9k + 1 = 3(9k^3 + 9k^2 + 3k) + 1$.
- when $n = 3k+2$, $n^3 = 27k^3 + 54k^2 + 36k + 8 = 3(9k^3 + 18k^2 + 12k + 2) + 2$.

In either case $3 \nmid n^3$.

(c) Using the previous results (or otherwise) show that $3^{1/3}$ is an irrational number.

Solution: Assume, to the contrary, that $3^{1/3}$ is rational. Hence we can write $3^{1/3} = a/b$ where $a, b \in \mathbb{Z}$ and further a, b have no common factors.
Now

$$3 = \frac{a^3}{b^3}$$
$$3b^3 = a^3$$

Hence $3 \mid a^3$ and so by the previous result we must have that $3 \mid a$. We can write $a = 3p$ with $p \in \mathbb{Z}$. Now

$$3b^3 = a^3 = 27p^3$$
$$b^3 = 9p^3$$

Hence $3 \mid b^3$ and so $3 \mid b$.
This gives a contradiction since 3 is a factor of both a, b. So $3^{1/3}$ is not rational.

10 marks 8. (a) Let G be a simple graph with 6 vertices with degrees 5,5,4,3,2,1. If G exists then draw it, otherwise explain why G does not exist.

Solution: No such graph exists. Let the vertices of the graph be denoted $v_1, \ldots, v_6$, and assume their degrees are $5, 5, 4, 3, 2, 1$ respectively. Then since v_1 has degree 5 it must be adjacent to all the other vertices. Similarly v_2 must be connected to all other vertices. Hence all other vertices v_3, v_4, v_5, v_6 must be adjacent to at least 2 other vertices. This contradicts G having a vertex of degree 1.

(b) Show that every connected graph on n vertices has at least $n-1$ edges.

Solution: We use induction on the number of vertices.

- Base case — one can quickly vertify that the statement is true for a connected graph on 1 vertex (it has no edges) and 2 vertices (it has at least one edge joining the two vertices).
- Inductive step — assume the statement is true for all connected graphs on n vertices or fewer. Let G be a connected graph on $n+1$ vertices. It suffices to show that G has at least n edges.

  Pick any vertex v of G. If that vertex has degree ℓ then when we delete v and all of the ℓ edges attached to it, we split G into at most ℓ connected components. It cannot be more than this otherwise v would need to be connected to more than ℓ different vertices.

  Each of these components contains $n_1, n_2, \ldots, n_ell$ vertices. By assumption each is connected with at most n vertices, so they contain at least $n_1 - 1, n_2 - 1, \ldots, n_\ell - 1$ edges. Thus the original graph G contains at least $(n_1 - 1) + (n_2 - 1) + \ldots (n_\ell - 1) + \ell = n_1 + n_2 + \cdots + n_\ell = n$ edges. Hence G contains at least n edges as required.

By induction the statement is true for all $n \in \mathbb{N}$.