Math 226 - Advanced Calculus I
December 2005

1. (a) Prove that the line given by the parametric equations $x = 1+4t$, $y = 2-t$, $z = -3t$, is parallel to the plane $2x + 5y + z = 4$.

We check that the direction vector of the line $(4, -1, -3)$ is perpendicular to the vector $\mathbf{n} = (2, 5, 1)$ normal to the plane: $(4, -1, -3) \cdot (2, 5, 1) = 8 - 5 - 3 = 0$.

(b) Find the distance between the plane and the line in (a).

Pick a point on the line, e.g. $P(1, 2, 0)$, and one in the plane, e.g. $Q(1, 0, 2)$. Then $\vec{QP} = (0, 2, -2)$. The distance from the line to the plane is equal to the scalar projection of $\vec{QP}$ on $\mathbf{n}$:

$$\frac{|(2,5,1)\cdot(0,2,-2)|}{\|(2,5,1)\|} = \frac{|0+10-2|}{\sqrt{4+25+1}} = \frac{8}{\sqrt{30}}.$$

2. Find all points on the surface $3x^2 - y^2 + 2z^2 = 1$ where the tangent plane is parallel to both of the vectors $(2, 2, -1)$ and $(4, 1, -5)$.

We find a vector perpendicular to $(2, 2, -1)$ and $(4, 1, -5)$:

$$(2,2,-1)\times(4,1,-5) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 4 & 1 & -5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -1 \\ 1 & -5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 4 & -5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = -9\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}.$$

So we need to find points on the surface where the normal vector to the surface is parallel to $(3, -2, 2)$. The normal vector at (x, y, z) is $(6x, -2y, 4z)$, or $(3x, -y, 2z)$ (divide by 2). Thus we should have for some t,

$$3x = 3t, -y = -2t, 2z = 2t,$$

i.e. $x = t, y = 2t, z = t$. If we plug this into the equation of the surface, we get

$$3t^2 - 4t^2 + 2t^2 = t^2 = 1,\ t = \pm 1.$$

This corresponds to two points, $(x, y, z) = (1, 2, 1)$ or $(-1, -2, -1)$.

3. (a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(x, y) = (1, 0)$, if $z = f(e^{x+2y}, \sin(xy), e^{x-y})$ and $f : \mathbf{R}^3 \to \mathbf{R}$ is a function of class C^1 such that $f(e, 0, e) = 3$ and $\nabla f(e, 0, e) = (3, -1, 2)$. (Use the Chain Rule).

By the Chain Rule, we have

$$\left.\frac{\partial z}{\partial x}\right|_{(1,0)} = 3e^{x+2y} - y\cos(xy) + 2e^{x-y}\Big|_{(1,0)} = 3e - 0 + 2e = 5e,$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,0)} = 3 \cdot 2e^{x+2y} - x\cos(xy) - 2e^{x-y}\Big|_{(1,0)} = 6e - 1 - 2e = 4e - 1.$$

(b) If $\mathbf{F}(x, y) = \begin{pmatrix} z \\ z^2 \end{pmatrix}$*, where* z *is as in (a), find* $D\mathbf{F}(1, 0)$*.*

We have, again by the Chain Rule,

$$\left.\frac{\partial(z^2)}{\partial x}\right|_{(1,0)} = 2z\left.\frac{\partial z}{\partial x}\right|_{(1,0)} = 2 \cdot 3 \cdot 5e = 30e,$$

$$\left.\frac{\partial(z^2)}{\partial y}\right|_{(1,0)} = 2z\left.\frac{\partial z}{\partial y}\right|_{(1,0)} = 2 \cdot 3 \cdot (4e - 1) = 24e - 6.$$

Hence

$$D\mathbf{F}(1, 0) = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial(z^2)}{\partial x} & \frac{\partial(z^2)}{\partial y} \end{pmatrix} = \begin{pmatrix} 5e & 4e - 1 \\ 30e & 24e - 6 \end{pmatrix}.$$

4. (a) Find the local maximum and minimum values and saddle points of the function $f(x, y) = x^4 + y^4 - 4xy + 6$*.*

We have

$$f_x = 4x^3 - 4y,\ f_y = 4y^3 - 4x,$$

$$f_{xx} = 12x^2,\ f_{yy} = 12y^2,\ f_{xy} = -4.$$

We first find critical points: if $f_x = f_y = 0$, then $x^3 = y$ and $y^3 = x$, so that $x^9 = y^3 = x$, $x = 0$ or $x^8 = 1$, $x = \pm 1$. We get three critical points: $(0, 0), (1, 1), (-1, -1)$. Now the second derivative test: $(0, 0)$ is a saddle point because

$$f_{xx}(0, 0) = 0, \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0,$$

$(1, 1)$ and $(-1, -1)$ are local minimizers because

$$f_{xx}(\pm 1, \pm 1) = 12, \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144 - 16 > 0.$$

Thus f has two local minima $f(1, 1) = f(-1, -1) = 4$ and one saddle point $f(0, 0) = 6$.

(b) Does the function in (a) have a global maximum or minimum? Explain why or why not.

Since $f(x, y) \to \infty$ as $\|(x, y)\| \to \infty$, there is no global maximum, and the two local minima at $(\pm 1, \pm 1)$ are in fact global minima.

5. The plane $x + 2y + z = 10$ *intersects the paraboloid* $z = x^2 + y^2$ *in an ellipse. Find the points on this ellipse which are nearest to and farthest from the origin.*

We need to find the critical points of $f(x, y, z) = x^2 + y^2 + z^2$ subject to constraints $g_1(x, y, z) = x+2y+z = 10$ and $g_2(x, y, z) = x^2+y^2-z = 0$. We use Lagrange multipliers. Since

$$\nabla f = (2x, 2y, 2z),\ \nabla g_1 = (1, 2, 1), \nabla g_2 = (2x, 2y, -1),$$

the critical points must satisfy for some λ_1, λ_2

$$2x = \lambda_1 + 2x\lambda_2,\ 2y = 2\lambda_1 + 2y\lambda_2,\ 2z = \lambda_1 - \lambda_2.$$

From the first two equations we have

$$2x(1 - \lambda_2) = \lambda_1,\ 2x(1 - \lambda_2) = 2\lambda_1.$$

Thus either $y = 2x$, or else $1 - \lambda_2 = \lambda_1 = 0$. In the second case we would have $2z = 0 - 1 = -1$, which contradicts the fact that $z = x^2 + y^2$ should be nonnegative. Therefore $y = 2x$. Plugging this into $g_1 = 10$ and $g_2 = 0$ we get

$$x + 4x + z = 5x + z = 10,\ x^2 + 4x^2 = 5x^2 = z.$$

Hence $10 = 5x + z = 5x + 5x^2$, $x^2 + x - 2 = 0$, $x = 1$ or -2. If $x = 1$, then $y = 2x = 2$ and $z = 5x^2 = 5$, and if $x = -2$ then $y = -4$ and $z = 20$. Clearly, $(1, 2, 5)$ will minimize the distance to the origin, and $(-2, -4, 20)$ will maximize it.

6. In each part of this problem, provide a precise definition of the word or phrase in boldface. Let

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Prove that f is **continuous** *at $(0, 0)$. (Hint: use polar coordinates.)*

f is continuous at $\mathbf{a}$ if the limit $\lim_{(x,y)\to\mathbf{a}} f(x, y)$ exists and is equal to $f(\mathbf{a})$. Here $\mathbf{a} = (0, 0)$. In polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have

$$f(x, y) = \frac{r^2 \cos\theta \sin\theta}{r} = r\cos\theta \sin\theta.$$

Thus $-r \leq f(x, y) \leq r$. As $(x, y) \to 0$, $r \to 0$, hence $\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$ as required.

(b) If $\mathbf{u}$ *is a unit vector, find the* **directional derivative** $D_{\mathbf{u}} f(0, 0)$ *directly from the definition.*

The directional derivative $D_{\mathbf{u}} f(\mathbf{a})$ is

$$D_{\mathbf{u}} f(\mathbf{a}) = \lim_{h\to 0} \frac{1}{h}(f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})).$$

If $\mathbf{u} = (u_1, u_2) = (\cos\theta, \sin\theta)$ is a unit vector, then by the first part,

$$D_{\mathbf{u}} f(0,0) = \lim_{h\to 0} \frac{1}{h}(f(h\cos\theta, h\sin\theta) - 0) = \lim_{h\to 0} \frac{h\cos\theta\sin\theta}{h} = \cos\theta\sin\theta = u_1 u_2.$$

*(c) Is f **differentiable** at $(0,0)$? Explain why or why not.*

f is differentiable at (a,b) if f_x and f_y exist at (a,b) and if

$$\lim_{(h,k)\to 0} \frac{1}{\|(h,k)\|}(f(a+h, b+k) - hf_x(a,b) - kf_y(a,b)) = 0.$$

In this example, $f_x(0,0) = f_y(0,0) = 0$, so differentiability of f at 0 would imply that $D_{\mathbf{u}} f(0,0) = 0$ for any unit vector $\mathbf{u}$. But this is not the case, by (b). Hence f is not differentiable at $(0,0)$.

7. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function of class C^1 such that

$$f(t\mathbf{x}) = t^a f(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{R}^n, t > 0$$

for some fixed $a \in \mathbf{R}$ (such functions are called homogeneous of degree a). Prove that

$$\mathbf{x} \cdot \nabla f(\mathbf{x}) = af(\mathbf{x}).$$

(Hint: for fixed $\mathbf{x}$, differentiate $f(t\mathbf{x})$ with respect to t.)

Fix $\mathbf{x}$. On the one hand, we have by the Chain Rule

$$\frac{d}{dt} f(t\mathbf{x}) = x_1 \frac{\partial f}{\partial x_1} f(t\mathbf{x}) + \ldots + x_n \frac{\partial f}{\partial x_n} f(t\mathbf{x}) = \mathbf{x} \cdot \nabla f(t\mathbf{x}),$$

on the other hand, using the homogeneity of f we also have

$$\frac{d}{dt} f(t\mathbf{x}) = \frac{d}{dt}(t^a f(\mathbf{x})) = at^{a-1} f(\mathbf{x}).$$

Hence $\mathbf{x} \cdot \nabla f(t\mathbf{x}) = at^{a-1} f(\mathbf{x})$. Setting now $t = 1$, we get $\mathbf{x} \cdot \nabla f(\mathbf{x}) = af(\mathbf{x})$, as required.

8. Evaluate the following integrals.

(a) $\int\int_D 3dA$, if D is the region bounded by the parabola $y^2 - x - 5 = 0$ and the line $x + 2y = 3$.

We first find the points where the parabola intersects the line: if $x = y^2 - 5$ and $x + 2y = 3$ then $y^2 - 5 + 2y - 3$, $y^2 + 2y - 8 = 0$, $y = 2, -4$. It will be more convenient to integrate in x first (draw a picture!):

$$\int\int_D 3dA = \int_{-4}^{2} \int_{y^2-5}^{3-2y} 3dxdy = \int_{-4}^{2} 3(3 - 2y - y^2 + 5)dy$$

$$= 3\int_{-4}^{2}(-y^2 - 2y + 8)dy = 3(-\frac{y^3}{3} - y^2 + 8y)\Big|_{-4}^{2} = 108.$$

(b) $$\int_0^1\int_{x^2}^1 x^3\sin(y^3)dydx = \int_0^1\int_0^{\sqrt{y}} x^3\sin(y^3)dxdy = \int_0^1 (\frac{x^4}{4}\sin(y^3)\Big|_{x=0}^{\sqrt{y}}dy$$

$$= \int_0^1 \frac{y^2}{4}\sin(y^3)dy = -\frac{\cos(y^3)}{12}\Big|_0^1 = \frac{-\cos(1)+1}{12}.$$

9. Let R be the solid region in $\mathbf{R}^3$ bounded by the planes $x = 0$, $y = 0$, $y = 4 - x$, and the surface $z = 4 - x^2$. Write $\int\int\int_R f(x,y,z)dV$ as iterated integrals where the order of integration is as indicated below (i.e. find the limits of integration).

Actually, this defines two *unbounded* regions in $\mathbf{R}^3$, one below the surface $z = 4 - x^2$, one above it. (I had intended to add the condition $z \geq 0$, but it was left out of the typed version by mistake.) For the region *below* the surface $z = 4 - x^2$, the solution is as indicated below.

(a)

$$\int_0^4\int_0^{4-x}\int_{-\infty}^{4-x^2} f(x,y,z)dzdydx$$

(b)

$$\int_{-\infty}^{4}\int_0^{\min(4,\sqrt{4-z})}\int_0^{4-x} f(x,y,z)dydxdz$$

$$= \int_{-\infty}^{-12}\int_0^4\int_0^{4-x} f(x,y,z)dydxdz + \int_{-12}^{4}\int_0^{\sqrt{4-z}}\int_0^{4-x} f(x,y,z)dydxdz.$$