MATH 226 MIDTERM 1: SOLUTIONS

October 5, 2005

1. *Find the closest distance between the sphere* $(x-3)^2+(y-1)^2+(z-2)^2 = 1$ *and the plane* $2x+y-2z+6=0$.

   We first find the distance from the center of the sphere $P(3,1,2)$ to the plane. Using a formula from class/textbook, this is
   $$\frac{2\cdot 3+1\cdot 1-2\cdot 2+6}{\sqrt{4+1+4}} = \frac{9}{3} = 3.$$
   (Alternatively, this distance can be found by picking a point Q in the plane and projecting $\vec{PQ}$ onto the normal vector to the plane.) To get the distance from the closest point on the sphere to the plane, we subtract the radius of the sphere from this: $3-1=2$.

2. *Find the parametric equation for the line which passes through the point* $(2,0,0)$ *and intersects the line* $x=t, y=t+1, z=2t-1$ *at a right angle.*

   Let $P(t)=(t,t+1,2t-1)$ be a point on the given line, and let Q be the point $(2,0,0)$. For the line through P and Q to be perpendicular to the given line, the vector $\vec{QP}=(t-2,t+1,2t-1)$ needs to be perpendicular to the direction vector $\mathbf{a}=(1,1,2)$ of that line:
   $$0=\mathbf{a}\cdot\vec{PQ}=(t-2)+(t+1)+2(2t-1)=6t-3,\ t=\frac{1}{2}.$$
   Hence P must be the point $(\frac{1}{2},\frac{3}{2},0)$. The direction vector of the new line is parallel to $\vec{QP}=(-\frac{3}{2},\frac{3}{2},0)$; take $(-1,1,0)$ to simplify the numbers. The new line has the parametric equation $(2,0,0)+t(-1,1,0)$, or equivalently $x=2-t, y=t, z=0$.

   (There is also an alternative solution which uses projections.)

3. *Find the area of the triangle in* $\mathbb{R}^3$ *with vertices* $(1,0,0)$, $(4,0,1)$, $(1,2,-1)$.

   Denote the vertices $(1,0,0)$, $(4,0,1)$, $(1,2,-1)$ by P,Q,R. Then the area is $\frac{1}{2}\|\vec{PQ}\times\vec{PR}\|$. We compute:
   $$\vec{PQ}\times\vec{PR}=(3,0,1)\times(0,2,-1)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 0 & 1\\ 0 & 2 & -1\end{vmatrix}=-2\mathbf{i}+3\mathbf{j}+6\mathbf{k}.$$
   Hence the area is $\frac{1}{2}\sqrt{4+9+36}=\frac{1}{2}\sqrt{49}=\frac{7}{2}$.

4. *Consider the solid in* $\mathbb{R}^3$ *shown in the picture below (i.e. the upper half of the solid ball of radius 1 centered at* $(0,0,1)$*). Describe the solid in Cartesian, cylindrical, and spherical coordinates.*

   In Cartesian coordinates, the solid is described by
   $$x^2+y^2+(z-1)^2\le 1,\ z\ge 1.$$

The first inequality describes the solid ball, and the second one says that we take its upper half. To convert this to cylindrical coordinates, we use that $x^2 + y^2 = r^2$:

$$r^2 + (z-1)^2 \leq 1, \ z \geq 1.$$

Finally, we convert this to spherical coordinates, using that $r = \rho \sin\phi$ and $z = \rho\cos\phi$. The first inequality becomes

$$(\rho\sin\phi)^2 + (\rho\cos\phi - 1)^2 \leq 1,$$

which simplifies to $\rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 \leq 1$, $\rho^2 - 2\rho\cos\phi \leq 0$, $\rho - 2\cos\phi \leq 0$. Together with the second inequality $z \geq 1$, we get

$$\rho - 2\cos\phi \leq 0, \ \rho\cos\phi \geq 1.$$

5. *Decide whether each of the sets below is open, closed, or neither:*

   (a) $\{(x,y) \in \mathbb{R}^2 : \ 0 \leq x < 1, \ 0 \leq y < 2\}$ – neither,

   (b) $\{(x,y) \in \mathbb{R}^2 : \ x + y < 2\}$ – open.

6. Let $f(x,y) = \sqrt{y - x^2}$.

   (a) *Find the domain and range of f. Is f one-one? Is it onto?*

   Domain: $y \geq x^2$, range: $[0, \infty)$. f is not one-one, e.g. $f(x,y) = f(-x,y)$ for any x, y, neither is it onto, because the range of f is not all of $\mathbf{R}$.

   (b) *Draw several level curves of $f(x,y)$, indicating the height c of each curve.*

   (c) *Sketch (roughly) the graph of $f(x,y)$.*