MATH 226 MIDTERM 2 – SOLUTIONS

November 2005

1. A bug is walking on a surface given by the equation $z = x^2 - 2xy + 3y^2$. At a certain time, its (x, y)-coordinates are $(3, 2)$.

(a) In what direction should the bug go if it wants to climb as steeply as possible?

It should go in the direction of the gradient of z, i.e. $(2x - 2y, -2x + 6y) = (2, 6)$ at $(3, 2)$.

(b) If the bug does go in that direction, what angle will its trajectory make with the xy-plane?

$$\tan^{-1} \sqrt{2^2 + 6^2} = \tan^{-1} \sqrt{40}.$$

2. Suppose that a function $f(x, y)$ is differentiable at the point $(1, 2)$, and that at that point the directional derivatives of f in the direction of vectors $(3, 4)$ and $(-4, 3)$ are 8 and 6, respectively. Find the equation of the plane tangent to the surface $z = f(x, y)$ at $(x, y) = (1, 2)$.

Let $f_x(1, 2) = A$, $f_y(1, 2) = B$. The unit vectors in the direction of $(3, 4)$ and $(-4, 3)$ are $(3/5, 4/5)$ and $(-4/5, 3/5)$. Hence we have the equations

$$\frac{3}{5}A + \frac{4}{5}B = 8, \ -\frac{4}{5}A + \frac{3}{5}B = 6,$$

or equivalently, $3A + 4B = 40$ and $-4A + 3B = 30$. Solving this yields $A = 0$, $B = 10$. Thus the tangent plane has the equation $z = f(1, 2) + 10(y - 2)$.

3. Do the following limits exist? Justify your answer. (A δ-ϵ proof is not needed, but you should give a reasonable argument.)

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x - y)^2}{x^2 + y^2}$ does not exist. If we approach $(0, 0)$ along the line $x = y$, we get $\displaystyle\lim_{x\to 0} 0 = 0$. But if we let $y = 0$, we get $\displaystyle\lim_{x\to 0} \frac{x^2}{x^2} = 1$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2 + y^2} = 0$: since $0 \le x^2 \le x^2 + y^2$, we have $0 \le \dfrac{x^4}{x^2 + y^2} \le x^2$, and that approaches 0 as $x \to 0$.

4. Find all points on the hyperboloid $4x^2 - y^2 - z^2 = 24$ where the tangent plane is parallel to the plane $8x + 3y - z = 2$.

We are looking for points (a, b, c) on the hyperboloid where the normal vector $(8a, -2b, -2c) = 2(4a, -b, -c)$ is parallel to $(8, 3, -1)$, i.e. $(4a, -b, -c) = (8t, 3t, -t)$ for some t. This gives $a = 2t, b = -3t, c = t$. We find all points on the hyperboloid which satisfy these equations:

$$4 \cdot 4t^2 - 9t^2 - t^2 = 6t^2 = 24, \ t^2 = 4, \ t = \pm 2.$$

We get two points: $(4, -6, 2)$ and $(-4, 6, -2)$.

5. (a) Let $f(x, y) = xy(x + 2y - 6)$. Find all critical points of f and classify them as maxima, minima, or saddle points.

We have $f(x, y) = x^2y + 2xy^2 - 6xy$, so that

$$f_x = 2xy + 2y^2 - 6y = 2y(x + y - 3), \ f_y = x^2 + 4xy - 6x = x(x + 4y - 6).$$

Suppose that $f_x(x,y) = f_y(x,y) = 0$. Then either $y = 0$ or $y = 3 - x$. If $y = 0$, then either $x = 0$ or $x - 6 = 0$. If $y = 3 - x$, then either $x = 0$ (hence $y = 3$) or $x + 12 - 4x - 6 = 6 - 3x = 0$, $x = 2$ (hence $y = 1$).

We thus have 4 critical points $(0,0)$, $(6,0)$, $(0,3)$, $(2,1)$. We now need to check the Hessian at these points. We have

$$f_{xx} = 2y,\ f_{xy} = 2x + 4y - 6,\ f_{yy} = 4x.$$

- At $(0,0)$, $f_{xx} = 0$, $\det(Hf) = \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = -36 < 0$. Thus f has a saddle point here.
- At $(0,3)$, $f_{xx} = 6 > 0$, $\det(Hf) = \begin{vmatrix} 6 & 6 \\ 6 & 0 \end{vmatrix} = -36 < 0$. Again, f has a saddle point here.
- At $(6,0)$, $f_{xx} = 0$, $\det(Hf) = \begin{vmatrix} 0 & 6 \\ 6 & 24 \end{vmatrix} = -36 < 0$. One more saddle point.
- At $(2,1)$, $f_{xx} = 2 > 0$, $\det(Hf) = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 4 > 0$. Hence f has a minimum here.

(b) For the same function $f(x,y)$ as in (a), find the global minimum and maximum of $f(x,y)$ on the triangle with vertices $(0,0)$, $(0,6)$, $(6,0)$.

By (a), f has one critical point $(2,1)$ inside the triangle. At that point, $f(2,1) = 2 \cdot 1(2 + 2 - 6) = -4$. On the two sides of the triangle where $x = 0$ or $y = 0$, we have $f(x,y) = 0$. If remains to check the third side where $x + y = 6$, ie. $x = 6 - y$. We need to find the global extrema of

$$f(6-y, y) = y(6-y)(6 - y + 2y - 6) = y^2(6-y) = 6y^2 - y^3$$

for $0 \leq y \leq 6$. At the endpoints, $f(6,0) = f(0,6) = 0$. We also have

$$\frac{df(6-y,y)}{dy} = 12y - 3y^2 = 3y(4-y).$$

This is 0 when $y = 0$ (already checked) or $y = 4$ (hence $x = 2$). At that point, $f(2,4) = 2 \cdot 4(2+8-6) = 32$. Overall, the global minimum is $f(2,1) = -4$, and the global maximum is $f(2,4) = 32$.