MATH 226 MIDTERM 1, FALL 2009: SOLUTIONS

1. Find the domain and the range of the function $\mathbf{f}(x,y) = \big(y, \dfrac{1}{x+y}, \sqrt{2x+y}\big)$.

The domain is $\{(x,y) \in \mathbf{R}^2 : \ x+y \neq 0, \ 2x+y \geq 0\}$. For the range, let u,v,w be the coordinates in the target space, then $u = y$, $v = \dfrac{1}{x+y}$, $w = \sqrt{2x+y}$. Thus $y = u$, $x+y = \frac{1}{v}$, $x = \frac{1}{v} - y = \frac{1}{v} - u$, and

$$w = \sqrt{\frac{2}{v} - 2u + u} = \sqrt{\frac{2}{v} - u}.$$

Thus the range of the function is $\{(u,v,w) \in \mathbf{R}^3 : \ w = \sqrt{\frac{2}{v} - u}\}$. (If you wish, you can now relabel the coordinates u,v,w as x,y,z, then the range is $\{(x,y,z) \in \mathbf{R}^3 : \ z = \sqrt{\frac{2}{y} - x}\}$.)

2. Find the parametric equation for the line which passes through the point $(4,5,-2)$ *and is perpendicular to the plane with parametric equations* $x = 1+3s-t$, $y = s+t$, $z = 7+2t$.

The plane is parallel to the vectors $(3,1,0)$ and $(-1,1,2)$, hence the direction vector (A,B,C) of the line needs to be perpendicular to both of them. This leads to a system of equations $3A+B=0$ and $-A+B+2C=0$. From the first equation we have $B=-3A$, and plugging this into the second one we get $-A-3A+2C=0$, $C=2A$. Thus we may take $(A,B,C)=(1,-3,2)$. The equation of the line is $(x,y,z) = (4,5,-2)+t(1,-3,2)$, or $x = 4+t$, $y = 5-3t$, $z = -2+2t$.

3. A surface in $\mathbb{R}^3$ *has the equation* $\rho^2 - 4\rho\sin\phi + 3 = 0$ *in spherical coordinates. Find its equation in cylindrical coordinates. Sketch the surface.*

We have $r = \rho\sin\theta$ and $\rho^2 = r^2+z^2$, hence our equation in cylindrical coordinates is $r^2+z^2-4r+3=0$. To see what kind of a surface this is, we rewrite the equation as $(r-2)^2+z^2=1$. This is the torus obtained by rotating the circle $(x-2)^2+z^3=1$ in the xz-plane around the z-axis.

4. Evaluate the following limit, or explain why it fails to exist.

We rewrite the limit in polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^3y - xy^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)}{r^2} = \lim_{r\to 0} r^2(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta).$$

This limit exists and is equal to 0, because $r^2 \to 0$ and $\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$ is always between -2 and 2.

5. Let

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

(a) Find $\frac{\partial f}{\partial x}(0,0)$ *and* $\frac{\partial f}{\partial y}(0,0)$.

We have $f(x,0) = \frac{x^3}{x^2} = x$ and $f(0,y) = \frac{-y^3}{y^2} = -y$, hence $\frac{\partial f}{\partial x}(0,0) = 1$ and $\frac{\partial f}{\partial y}(0,0) = -1$.

(b) Is f differentiable at $(0,0)$? Explain your answer.
For f to be differentiable at $(0,0)$, we must have

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{\sqrt{x^2+y^2}} = 0,$$

where $h(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = x - y$. We rewrite the above limit as

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^3-y^3}{x^2+y^2} - (x-y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^3 - y^3 - x^3 - xy^2 + x^2y + y^3}{(x^2+y^2)^{3/2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2y - xy^2}{(x^2+y^2)^{3/2}}.$$

But if we let $x = -y$, then this limit is

$$\lim_{y\to 0} \frac{2y^3}{(2y^2)^{3/2}} = \lim_{y\to 0} \frac{2y^3}{2^{2/3}y^3} = 2^{1/3},$$

which is different from 0. Hence f fails to be differentiable at $(0,0)$.

6. Find all values of (a,b) such that the tangent plane to the hyperboloid $z = 4x^2 - y^2$ at $(a,b,4a^2-b^2)$ contains the line with parametric equations $x = t+1$, $y = 2t$, $z = -4t+9$.

Let $f(x,y) = 4x^2 - y^2$, then $f_x(a,b) = 8a$ and $f_y(a,b) = -2b$. Hence the tangent plane at $(a,b,4a^2-b^2)$ has the equation

$$z = 4a^2 - b^2 + 8a(x-a) - 2b(y-b) = 8ax - 2by - 4a^2 + b^2.$$

This contains the line in question if all points on the line satisfy the equation of the plane, i.e. for all $t \in \mathbf{R}$ we must have

$$-4t + 9 = 8a(t+1) - 2b(2t) - 4a^2 + b^2, \quad (8a - 4b + 4)t + 8a - 4a^2 + b^2 - 9 = 0.$$

Thus $8a - 4b + 4 = 0$, so that $b = 2a + 1$. We also must have $0 = 8a - 4a^2 + b^2 - 9 = 8a - 4a^2 + (2a+1)^2 - 9 = 12a - 8$, so that $a = 2/3$ and $b = 7/3$.