MATH 226 MIDTERM 2, FALL 2009: SOLUTIONS

1. (a) Let $f(x,y)$ be a C^2 function with values in $\mathbf{R}$. Write out the general formula for the first and second order Taylor polynomials of $f(x,y)$ at a point (a,b).

$$p_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b),$$

$$p_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2} f_{xx}(a,b)(x-a)^2$$
$$+ f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2} f_{yy}(a,b)(y-b)^2$$

(b) Find the second order Taylor polynomial of the function $f(x,y) = \sin(x+y^2)$ at $(\pi, 0)$.

$$f(\pi,0) = \sin\pi = 0,$$
$$f_x(x,y) = \cos(x+y^2),\ f_x(\pi,0) = \cos\pi = -1,$$
$$f_y(x,y) = 2y\cos(x+y^2),\ f_x(\pi,0) = 0,$$
$$f_{xx}(x,y) = -\sin(x+y^2),\ f_{xx}(\pi,0) = -\sin\pi = 0,$$
$$f_{xy}(x,y) = -2y\sin(x+y^2),\ f_{xy}(\pi,0) = 0,$$
$$f_{yy}(x,y) = -4y^2\sin(x+y^2) + 2\cos(x+y^2),\ f_{yy}(\pi,0) = 2\cos\pi = -2,$$
$$p_2(x,y) = -(x-\pi) - y^2.$$

2. (a) Let

$$f(x,y) = \begin{cases} \dfrac{x^5}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Find the general formula for $D_{\mathbf{u}} f(0,0)$, where $\mathbf{u} = (u_1, u_2)$ is a unit vector, in terms of u_1, u_2.

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to 0} \frac{1}{t}(f(t\mathbf{u}) - f(0,0)) = \frac{1}{t}\Big(\frac{t^5 u_1^5}{t^4 u_1^4 + t^4 u_2^4} - 0\Big) = \frac{1}{t}\frac{t u_1^5}{u_1^4 + u_2^4} = \frac{t u_1^5}{u_1^4 + u_2^4}.$$

(b) Suppose that $f: \mathbf{R}^2 \to \mathbf{R}$ is a function such that $D_{\mathbf{u}} f(0,0) = u_1^2 - u_2^2$ for any unit vector $\mathbf{u} = (u_1, u_2)$. Can f differentiable be at $(0,0)$? Explain your answer.

We have $f_x(0,0) = D_{\mathbf{i}} f(0,0) = 1^2 - 0^2 = 1$ and $f_y(0,0) = D_{\mathbf{j}} f(0,0) = 0^2 - 1^2 = -1$, hence $\nabla f(0,0) = (1,-1)$. If f were differentiable, we would have $D_{\mathbf{u}} f(0,0) = \nabla f(0,0)\cdot\mathbf{u} = u_1 - u_2$ for all $\mathbf{u}$. But this is not consistent with $D_{\mathbf{u}} f(0,0) = u_1^2 - u_2^2$: for example when $\mathbf{u} = -\mathbf{i}$, the first formula gives $D_{-\mathbf{i}} f(0,0) = -1 - 0 = -1$ and the second one gives $D_{-\mathbf{i}} f(0,0) = 1^2 - 0^2 = 1$. Therefore f cannot be differentiable.

3. (a) Let $w = f(a_1 x + a_2 y + a_3 z, b_1 x + b_2 y + b_3 z)$, where $f: \mathbf{R}^2 \to \mathbf{R}$ is a C^1 function. Prove that

$$c_1 \frac{\partial w}{\partial x} + c_2 \frac{\partial w}{\partial y} + c_3 \frac{\partial w}{\partial z} = 0$$

for any vector (c_1, c_2, c_3) orthogonal to both (a_1, a_2, a_3) and (b_1, b_2, b_3).

We have $w = f(u,v)$, where $u = a_1x + a_2y + a_3z$ and $v = b_1x + b_2y + b_3z$. By the Chain Rule,

$$c_1\frac{\partial w}{\partial x} + c_2\frac{\partial w}{\partial y} + c_3\frac{\partial w}{\partial z} = c_1(f_ua_1 + f_vb_1) + c_2(f_ua_2 + f_vb_2) + c_3(f_ua_3 + f_vb_3)$$

$$= (c_1a_1 + c_2a_2 + c_3a_3)f_u + (c_1b_1 + c_2b_2 + c_3b_3)f_v = (\mathbf{c}\cdot\mathbf{a})f_u + (\mathbf{c}\cdot\mathbf{b})f_v = 0.$$

(b) Assume that $f : \mathbf{R}^2 \to \mathbf{R}$ is a C^1 function and that the point $(1,2,-3)$ lies on the surface $f(2x-y+z, x-z) = 10$. What condition should f satisfy so that the equation $f(2x-y+z, x-z) = 10$ could be solved for z as a differentiable function of x and y near the point $(1,2,-3)$? (Use the Implicit Function Theorem.)

Bu the Implicit Function Theorem, we can solve for z as required if $\partial_z f(2x-y+z, x-z) \neq 0$ at $(1,2,-3)$. As in (a), we have $\partial_z f(2x-y+z, x-z) = f_u - f_v$. Also, at $(1,2,-3)$ we have $u = 2-2-3 = -3$ and $v = 1+3 = 4$. Hence the needed condition is $f_u(-3,4) - f_v(-3,4) \neq 0$.

4. Find the minimum and maximum values of the function $f(x,y) = 4x - 2xy + y^2$ on the square $\{(x,y) : 0 \le x \le 2,\ 0 \le y \le 2\}$.

We first look for critical points inside the region. We have $f_x = 4 - 2y$ and $f_y = -2x + 2y$, hence if $f_x = f_y = 0$, then $y = 2$ and $x = 2$. At the critical point, $f(2,2) = 8 - 8 + 4 = 4$.

Next, we look for possible minima and maxima on the boundary:

- $f(x,0) = 4x$, minimum value on $[0,2]$ is $f(0,0) = 0$, maximum value is $f(2,0) = 8$;
- $f(x,2) = 4x - 4x + 4 = 4$;
- $f(0,y) = y^2$, minimum value on $[0,2]$ is $f(0,0) = 0$, maximum value is $f(0,2) = 4$;
- $f(2,y) = 8 - 4y + y^2$. To find its extrema on $[0,2]$, we look for critical points: $-4 + 2y = 0$, $y = 2$. We have already evaluated $f(2,2) = 4$ and $f(2,0) = 8$.

Thus the smallest value is $f(0,0) = 0$ and the largest value is $f(2,0) = 8$.