*This midterm has **5 questions** on **5 pages**, for a total of 40 points.*

Duration: 50 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (Last, First, All middle names): ______________________

Student-No: ______________________

Signature: ______________________

Question:	1	2	3	4	5	Total
Points:	10	6	6	8	10	40
Score:						

10 marks 1. (a) Let

$$f(x,y) = \begin{cases} \dfrac{x^5}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Find the general formula for $D_{\mathbf{u}}f(0,0)$, where $\mathbf{u} = (u_1, u_2)$ is a unit vector, in terms of u_1, u_2.

Solution:

$$D_{\mathbf{u}}f(0,0) = \lim_{t\to 0} \frac{1}{t}(f(t\mathbf{u}) - f(0,0)) = \frac{1}{t}\left(\frac{t^5u_1^5}{t^4u_1^4 + t^4u_2^4} - 0\right) = \frac{1}{t}\frac{tu_1^5}{u_1^4+u_2^4} = \frac{u_1^5}{u_1^4+u_2^4}.$$

(b) Is there a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D_{\mathbf{u}}f(0,0) = u_1^2 - u_2^2$ for every unit vector $\mathbf{u} = (u_1, u_2)$? If yes, find it. If no, explain why.

Solution:
If there were such a function, we would have $f_x(0,0) = D_{\mathbf{i}}f(0,0) = 1^2 - 0^2 = 1$ and $f_y(0,0) = D_{\mathbf{j}}f(0,0) = 0^2 - 1^2 = -1$, hence $\nabla f(0,0) = (1,-1)$. We would also have $D_{\mathbf{u}}f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = u_1 - u_2$ for all $\mathbf{u}$. But this is not consistent with $D_{\mathbf{u}}f(0,0) = u_1^2 - u_2^2$: for example when $\mathbf{u} = -\mathbf{i}$, the first formula gives $D_{-\mathbf{i}}f(0,0) = -1 - 0 = -1$ and the second one gives $D_{-\mathbf{i}}f(0,0) = 1^2 - 0^2 = 1$. there is no such function.

6 marks

2. Let $w = f(a_1x + a_2y + a_3z, b_1x + b_2y + b_3z)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous first order partial derivatives. Prove that

$$c_1\frac{\partial w}{\partial x} + c_2\frac{\partial w}{\partial y} + c_3\frac{\partial w}{\partial z} = 0$$

for any vector (c_1, c_2, c_3) orthogonal to both (a_1, a_2, a_3) and (b_1, b_2, b_3).

Solution:

We have $w = f(u, v)$, where $u = a_1x + a_2y + a_3z$ and $v = b_1x + b_2y + b_3z$. By the Chain Rule,

$$c_1\frac{\partial w}{\partial x} + c_2\frac{\partial w}{\partial y} + c_3\frac{\partial w}{\partial z} = c_1(f_ua_1 + f_vb_1) + c_2(f_ua_2 + f_vb_2) + c_3(f_ua_3 + f_vb_3)$$

$$= (c_1a_1 + c_2a_2 + c_3a_3)f_u + (c_1b_1 + c_2b_2 + c_3b_3)f_v = (\mathbf{c}\cdot\mathbf{a})f_u + (\mathbf{c}\cdot\mathbf{b})f_v = 0.$$

6 marks

3. Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function and that the point $(1, 2, -3)$ lies on the surface $f(2x - y + z, x - z) = 10$. What condition should f satisfy so that the equation $f(2x - y + z, x - z) = 10$ could be solved for z as a differentiable function of x and y near the point $(1, 2, -3)$? (Use the Implicit Function Theorem.)

Solution: By the Implicit Function Theorem, we can solve for z as required if $\partial_z f(2x-y+z, x-z) \neq 0$ at $(1, 2, -3)$. As in (a), we have $\partial_z f(2x-y+z, x-z) = f_u - f_v$. Also, at $(1, 2, -3)$ we have $u = 2 - 2 - 3 = -3$ and $v = 1 + 3 = 4$. Hence the needed condition is $f_u(-3, 4) - f_v(-3, 4) \neq 0$.

8 marks 4. Find the second order Taylor polynomial of the function $f(x, y) = \sin(x + y^2)$ at $(\pi, 0)$.

Solution:

$$f(\pi, 0) = \sin \pi = 0,$$
$$f_x(x, y) = \cos(x + y^2),\ \ f_x(\pi, 0) = \cos \pi = -1,$$
$$f_y(x, y) = 2y \cos(x + y^2),\ \ f_x(\pi, 0) = 0,$$
$$f_{xx}(x, y) = -\sin(x + y^2),\ \ f_{xx}(\pi, 0) = -\sin \pi = 0,$$
$$f_{xy}(x, y) = -2y \sin(x + y^2),\ \ f_{xy}(\pi, 0) = 0,$$
$$f_{yy}(x, y) = -4y^2 \sin(x + y^2) + 2 \cos(x + y^2),\ \ f_{yy}(\pi, 0) = 2 \cos \pi = -2,$$
$$p_2(x, y) = -(x - \pi) - y^2.$$

10 marks 5. Find the largest and smallest values of the function $f(x,y) = 4x - 2xy + y^2$ on the square $\{(x,y) : \ 0 \le x \le 2, \ 0 \le y \le 2\}$.

Solution: We first look for critical points inside the region. We have $f_x = 4 - 2y$ and $f_y = -2x + 2y$, hence if $f_x = f_y = 0$, then $y = 2$ and $x = 2$. At the critical point, $f(2,2) = 8 - 8 + 4 = 4$.

Next, we look for possible minima and maxima on the boundary:

- $f(x,0) = 4x$, minimum value on $[0,2]$ is $f(0,0) = 0$, maximum value is $f(2,0) = 8$;
- $f(x,2) = 4x - 4x + 4 = 4$;
- $f(0,y) = y^2$, minimum value on $[0,2]$ is $f(0,0) = 0$, maximum value is $f(0,2) = 4$;
- $f(2,y) = 8 - 4y + y^2$. To find its extrema on $[0,2]$, we look for critical points: $-4 + 2y = 0$, $y = 2$. We have already evaluated $f(2,2) = 4$ and $f(2,0) = 8$.

Thus the smallest value is $f(0,0) = 0$ and the largest value is $f(2,0) = 8$.