This final exam has ***8 questions*** *on* ***10 pages****, for a total of 100 marks.*
Duration: 2 hours 30 minutes

Name (last, first, all middle names): ____________________

Student-No: ____________________ Course Section: ____________

Signature: ____________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	12	16	10	16	10	16	10	100
Score:									

Please read the following points carefully before starting to write.

- In all questions, give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Write clearly and legibly, in complete sentences. Make sure that the logic of your argument is clear.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space.

10 marks 1. The plane $x + 2y + 4z = 9$ intersects the cone $z^2 = x^2 + 2y^2$ in an ellipse that contains the point $(1, -2, 3)$. Find the parametric equations of the line tangent to that ellipse at that point.

Solution:

The line tangent to the ellipse must lie in the plane $x+2y+4z = 9$, therefore it must be perpendicular to its normal vector $\mathbf{n}_1 = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. It must also be tangent to the cone $z^2 = x^2 + 2y^2$. A normal vector to the cone is given by $\nabla(x^2 + 2y^2 - z^2) = \langle 2x, 4y, -2z\rangle = 2\langle x, 2y, -z\rangle = 2\langle 1, -4, -3\rangle$ at $(1, -2, 3)$. To simplify calculations, we will use the normal vector $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$. The line in question is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, so we can find a vector parallel to it from the cross-product

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 4 \\ 1 & -4 & -3 \end{vmatrix} = (-6+16)\mathbf{i} - (-3-4)\mathbf{j} + (-4-2)\mathbf{k} = 10\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}.$$

The parametric equations of the line are $x = 1 + 10t$, $y = -2 + 7t$, $z = 3 - 6t$.

$\boxed{\text{12 marks}}$ 2. Let $f(x, y) = xe^{x-2y}$.

(a) Find the directional derivative of f at $(x, y) = (2, 1)$ in the direction of the vector $3\mathbf{i} - \mathbf{j}$.

Solution: We have

$$f_1(x, y) = e^{x-2y} + xe^{x-2y}, \quad f_1(2, 1) = e^0 + 2e^0 = 3,$$
$$f_2(x, y) = -2xe^{x-2y}, \quad f_1(2, 1) = -4e^0 = -4,$$

so that $\nabla f(2, 1) = 3\mathbf{i} - 4\mathbf{j}$. We also need a unit vector in the direction of $3\mathbf{i} - \mathbf{j}$:

$$\mathbf{v} = \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{3^2 + 1^2}} = \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{10}}.$$

Therefore

$$D_{\mathbf{v}} f(2, 1) = (3\mathbf{i} - 4\mathbf{j}) \cdot \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{10}} = \frac{3 \cdot 3 + (-4)(-1)}{\sqrt{10}} = \frac{13}{\sqrt{10}}.$$

(b) Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}} f(2, 1) = 0$.

Solution: The vector $\mathbf{u}$ should be perpendicular to $\nabla f(2, 1) = 3\mathbf{i} - 4\mathbf{j}$, and should have length 1. We can take

$$\mathbf{u} = \frac{4\mathbf{i} + 3\mathbf{j}}{\sqrt{4^2 + 3^2}} = \frac{4\mathbf{i} + 3\mathbf{j}}{5}.$$

16 marks 3. Let $f(x, y) = \frac{y}{x} + \frac{1}{y} + x$.

(a) (6 marks) This function has exactly one critical point. Find it.

Solution: We have

$$f_1(x, y) = -\frac{y}{x^2} + 1, \quad f_2(x, y) = \frac{1}{x} - \frac{1}{y^2}.$$

For $f_1 = f_2 = 0$, we need $1 = y/x^2$, $y = x^2$, and $\frac{1}{x} = \frac{1}{y^2}$, so that $x = y^2 = (x^2)^2 = x^4$. We can't have $x = 0$ since then f would not be defined, so we must have $x^3 = 1$, $x = 1$, and $y = 1^2 = 1$. The critical point is at $(1, 1)$.

(b) (6 marks) Find the second order Taylor polynomial of $f(x, y)$ at the critical point found in (a).

Solution: We have $f(1, 1) = 1 + 1 + 1 = 3$, and

$$f_{11}(x, y) = \frac{2y}{x^3}, \quad f_{12}(x, y) = f_{21}(x, y) = -\frac{1}{x^2}, \quad f_{22}(x, y) = \frac{2}{y^3}.$$

Therefore $f_{11}(1, 1) = 2$, $f_{12}(1, 1) = f_{21}(1, 1) = -1$, and $f_{22}(1, 1) = 2$. The Taylor polynomial is

$$p_2(x, y) = 3 + \frac{1}{2}\Big(2(x-1)^2 - 2(x-1)(y-1) + 2(y-1)^2\Big).$$

(c) (4 marks) Does $f(x, y)$ have a minimum, maximum, or a saddle point at this critical point?

Solution:

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0, \quad f_{11}(1, 1) = 2 > 0,$$

so by the second derivative test, f has a local minimum at $(1, 1)$.

10 marks 4. Find the largest and smallest values of the function $f(x, y, z) = x$ subject to the constraints $x^2 + y^2 + z^2 = 120$ and $x + 2y - z = 0$.

Solution:

Omitted.

16 marks 5. Evaluate the double integrals. Use any method you like.

(a) (8 marks) $\iint_D y\,dA$, where D is the triangle in the xy-plane with vertices $(0,0)$, $(1,2)$, $(3,0)$.

Solution:

$$\begin{aligned}\int_0^2 \int_{y/2}^{3-y} y\,dx\,dy &= \int_0^2 y(3-y-\frac{y}{2})\,dy\\ &= \int_0^2 (3y-\frac{3y^2}{2})\,dy = \frac{3}{2}y^2 - \frac{y^3}{2}\Big|_0^2\\ &= \frac{3}{2}\cdot 4 - \frac{8}{2} = 6-4 = 2.\end{aligned}$$

(b) (8 marks) $\iint_{\mathbb{R}^2} \dfrac{dA}{(4+x^2+y^2)^5}$ (note that this is an improper integral, so the answer might be that the integral is divergent)

Solution:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\infty} \frac{r\,dr\,d\theta}{(4+r^2)^5} &= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2}(4+r^2)^{-5}(2r)\,dr\,d\theta\\ &= \int_0^{2\pi} \frac{-1}{8}(4+r^2)^{-4}\Big|_0^{\infty}\,d\theta = \int_0^{2\pi} \frac{1}{8}4^{-4}\,d\theta = \frac{2\pi}{8}4^{-4} = \frac{\pi}{4^5}.\end{aligned}$$

10 marks

6. Evaluate the triple integral $\iiint_D e^z\,dV$, where D is the tetrahedron bounded by the coordinate planes and the plane $3x+2y+z=6$.

Solution:

$$\int_0^2\int_0^{3-\frac{3x}{2}}\int_0^{6-3x-2y} e^z\,dz\,dy\,dx = \int_0^2\int_0^{3-\frac{3x}{2}} e^z\Big|_{z=0}^{6-3x-2y}\,dy\,dx$$

$$= \int_0^2\int_0^{3-\frac{3x}{2}} (e^{6-3x-2y}-1)\,dy\,dx = \int_0^2 (-\frac{1}{2}e^{6-3x-2y}-y)\Big|_{y=0}^{3-\frac{3x}{2}}\,dx$$

$$= \int_0^2 \Big(-\frac{1}{2}(e^{6-3x-(6-3x)}-e^{6-3x}) - (3-\frac{3x}{2})\Big)\,dx$$

$$= \int_0^2 \Big(-\frac{1}{2}+\frac{1}{2}e^{6-3x}-3+\frac{3x}{2}\Big)\,dx = \int_0^2 \Big(-\frac{7}{2}+\frac{1}{2}e^{6-3x}+\frac{3x}{2}\Big)\,dx$$

$$= \Big(-\frac{7x}{2}-\frac{1}{6}e^{6-3x}+\frac{3x^2}{4}\Big)\Big|_0^2 = -7-\frac{1}{6}(1-e^6)+3 = \frac{e^6-25}{6}.$$

16 marks

7. Let

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3 + 2xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (8 marks) Find the partial derivatives $f_1(0,0)$, $f_2(0,0)$.

Solution: We have $f(x,0) = \frac{x^3}{x^2} = x$ for $x \neq 0$, and $f(0,y) = -\frac{y^3}{y^2} = -y$ for $y \neq 0$. SInce these formulas match $f(0,0) = 0$, we have $f(x,0) = x$ and $f(0,y) = -y$ for all x, y. Therefore $f_1(0,0) = 1$ and $f_2(0,0) = -1$.

(b) (8 marks) If $f(x,y)$ differentiable at $(0,0)$? Prove your answer.

Solution:
Let $x = y$, then $f(x,x) = \frac{2x^2}{2x^2} = 1$. As $x \to 0$, this does not approach $f(0,0) = 1$. This means that f is not continuous at $(0,0)$ and therefore not differentiable at $(0,0)$.

10 marks 8. The quantities u, v are defined as functions of x, y so that they satisfy the equations $x = u\cos(u+v)$, $y = v\sin u$. Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ at the point where $u = \pi/3$, $v = 0$. (Hint: $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, $\cos(\pi/3) = \frac{1}{2}$.)

Solution:

Differentiating the equations $x = u\cos(u+v)$, $y = v\sin u$ in x, we get

$$\begin{aligned} 1 &= u_x\cos(u+v) - u\sin(u+v)\cdot(u_x+v_x), \\ 0 &= v_x\sin u + v\cos u\cdot u_x. \end{aligned}$$

At $u = \pi/3$, $v = 0$, we get

$$\begin{aligned} 1 &= u_x\cos(\pi/3) - \frac{\pi}{3}\sin(\pi/3)\cdot(u_x+v_x) = \frac{u_x}{2} - \frac{\pi\sqrt{3}}{6}(u_x+v_x), \\ 0 &= v_x\sin(\pi/3) = \frac{\sqrt{3}}{2}v_x. \end{aligned}$$

Hence $v_x = 0$, and

$$u_x = \frac{1}{\frac{1}{2} - \frac{\pi\sqrt{3}}{6}} = \frac{6}{3 - \pi\sqrt{3}}.$$