

*This final exam has **8 questions** on **10 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Name (last, first, all middle names): _____

Student-No: _____ Course Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	12	16	10	16	10	16	10	100
Score:									

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Please read the following points carefully before starting to write.

- In all questions, give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Write clearly and legibly, in complete sentences. Make sure that the logic of your argument is clear.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space.

10 marks

1. The plane $x + 2y + 4z = 9$ intersects the cone $z^2 = x^2 + 2y^2$ in an ellipse that contains the point $(1, -2, 3)$. Find the parametric equations of the line tangent to that ellipse at that point.

Solution:

The line tangent to the ellipse must lie in the plane $x + 2y + 4z = 9$, therefore it must be perpendicular to its normal vector $\mathbf{n}_1 = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. It must also be tangent to the cone $z^2 = x^2 + 2y^2$. A normal vector to the cone is given by $\nabla(x^2 + 2y^2 - z^2) = \langle 2x, 4y, -2z \rangle = 2\langle x, 2y, -z \rangle = 2\langle 1, -4, -3 \rangle$ at $(1, -2, 3)$. To simplify calculations, we will use the normal vector $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$. The line in question is perpendicular to both \mathbf{n}_1 and \mathbf{n}_2 , so we can find a vector parallel to it from the cross-product

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 4 \\ 1 & -4 & -3 \end{vmatrix} = (-6 + 16)\mathbf{i} - (-3 - 4)\mathbf{j} + (-4 - 2)\mathbf{k} = 10\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}.$$

The parametric equations of the line are $x = 1 + 10t$, $y = -2 + 7t$, $z = 3 - 6t$.

12 marks

2. Let $f(x, y) = xe^{x-2y}$.

- (a) Find the directional derivative of
- f
- at
- $(x, y) = (2, 1)$
- in the direction of the vector
- $3\mathbf{i} - \mathbf{j}$
- .

Solution: We have

$$f_1(x, y) = e^{x-2y} + xe^{x-2y}, \quad f_1(2, 1) = e^0 + 2e^0 = 3,$$

$$f_2(x, y) = -2xe^{x-2y}, \quad f_2(2, 1) = -4e^0 = -4,$$

so that $\nabla f(2, 1) = 3\mathbf{i} - 4\mathbf{j}$. We also need a unit vector in the direction of $3\mathbf{i} - \mathbf{j}$:

$$\mathbf{v} = \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{3^2 + 1^2}} = \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{10}}.$$

Therefore

$$D_{\mathbf{v}}f(2, 1) = (3\mathbf{i} - 4\mathbf{j}) \cdot \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{10}} = \frac{3 \cdot 3 + (-4)(-1)}{\sqrt{10}} = \frac{13}{\sqrt{10}}.$$

- (b) Find a unit vector
- \mathbf{u}
- such that
- $D_{\mathbf{u}}f(2, 1) = 0$
- .

Solution: The vector \mathbf{u} should be perpendicular to $\nabla f(2, 1) = 3\mathbf{i} - 4\mathbf{j}$, and should have length 1. We can take

$$\mathbf{u} = \frac{4\mathbf{i} + 3\mathbf{j}}{\sqrt{4^2 + 3^2}} = \frac{4\mathbf{i} + 3\mathbf{j}}{5}.$$

16 marks

3. Let $f(x, y) = \frac{y}{x} + \frac{1}{y} + x$.

- (a) (6 marks) This function has exactly one critical point. Find it.

Solution: We have

$$f_1(x, y) = -\frac{y}{x^2} + 1, \quad f_2(x, y) = \frac{1}{x} - \frac{1}{y^2}.$$

For $f_1 = f_2 = 0$, we need $1 = y/x^2$, $y = x^2$, and $\frac{1}{x} = \frac{1}{y^2}$, so that $x = y^2 = (x^2)^2 = x^4$. We can't have $x = 0$ since then f would not be defined, so we must have $x^3 = 1$, $x = 1$, and $y = 1^2 = 1$. The critical point is at $(1, 1)$.

- (b) (6 marks) Find the second order Taylor polynomial of $f(x, y)$ at the critical point found in (a).

Solution: We have $f(1, 1) = 1 + 1 + 1 = 3$, and

$$f_{11}(x, y) = \frac{2y}{x^3}, \quad f_{12}(x, y) = f_{21}(x, y) = -\frac{1}{x^2}, \quad f_{22}(x, y) = \frac{2}{y^3}.$$

Therefore $f_{11}(1, 1) = 2$, $f_{12}(1, 1) = f_{21}(1, 1) = -1$, and $f_{22}(1, 1) = 2$. The Taylor polynomial is

$$p_2(x, y) = 3 + \frac{1}{2} \left(2(x-1)^2 - 2(x-1)(y-1) + 2(y-1)^2 \right).$$

- (c) (4 marks) Does $f(x, y)$ have a minimum, maximum, or a saddle point at this critical point?

Solution:

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0, \quad f_{11}(1, 1) = 2 > 0,$$

so by the second derivative test, f has a local minimum at $(1, 1)$.

10 marks

4. Find the largest and smallest values of the function $f(x, y, z) = x$ subject to the constraints $x^2 + y^2 + z^2 = 120$ and $x + 2y - z = 0$.

Solution:

Omitted.

16 marks

5. Evaluate the double integrals. Use any method you like.

- (a) (8 marks) $\iint_D y \, dA$, where D is the triangle in the xy -plane with vertices $(0, 0)$, $(1, 2)$, $(3, 0)$.

Solution:

$$\begin{aligned} \int_0^2 \int_{y/2}^{3-y} y \, dx \, dy &= \int_0^2 y(3 - y - \frac{y}{2}) \, dy \\ &= \int_0^2 (3y - \frac{3y^2}{2}) \, dy = \frac{3}{2}y^2 - \frac{y^3}{2} \Big|_0^2 \\ &= \frac{3}{2} \cdot 4 - \frac{8}{2} = 6 - 4 = 2. \end{aligned}$$

- (b) (8 marks) $\iint_{\mathbb{R}^2} \frac{dA}{(4 + x^2 + y^2)^5}$ (note that this is an improper integral, so the answer might be that the integral is divergent)

Solution:

$$\begin{aligned} \int_0^{2\pi} \int_0^\infty \frac{r \, dr \, d\theta}{(4 + r^2)^5} &= \int_0^{2\pi} \int_0^\infty \frac{1}{2}(4 + r^2)^{-5}(2r) \, dr \, d\theta \\ &= \int_0^{2\pi} \frac{-1}{8}(4 + r^2)^{-4} \Big|_0^\infty \, d\theta = \int_0^{2\pi} \frac{1}{8}4^{-4} \, d\theta = \frac{2\pi}{8}4^{-4} = \frac{\pi}{4^5}. \end{aligned}$$

10 marks

6. Evaluate the triple integral $\iiint_D e^z dV$, where D is the tetrahedron bounded by the coordinate planes and the plane $3x + 2y + z = 6$.

Solution:

$$\begin{aligned} \int_0^2 \int_0^{3-\frac{3x}{2}} \int_0^{6-3x-2y} e^z dz dy dx &= \int_0^2 \int_0^{3-\frac{3x}{2}} e^z \Big|_{z=0}^{6-3x-2y} dy dx \\ &= \int_0^2 \int_0^{3-\frac{3x}{2}} (e^{6-3x-2y} - 1) dy dx = \int_0^2 \left(-\frac{1}{2} e^{6-3x-2y} - y \right) \Big|_{y=0}^{3-\frac{3x}{2}} dx \\ &= \int_0^2 \left(-\frac{1}{2} (e^{6-3x-(6-3x)} - e^{6-3x}) - \left(3 - \frac{3x}{2} \right) \right) dx \\ &= \int_0^2 \left(-\frac{1}{2} + \frac{1}{2} e^{6-3x} - 3 + \frac{3x}{2} \right) dx = \int_0^2 \left(-\frac{7}{2} + \frac{1}{2} e^{6-3x} + \frac{3x}{2} \right) dx \\ &= \left(-\frac{7x}{2} - \frac{1}{6} e^{6-3x} + \frac{3x^2}{4} \right) \Big|_0^2 = -7 - \frac{1}{6} (1 - e^6) + 3 = \frac{e^6 - 25}{6}. \end{aligned}$$

16 marks

7. Let

$$f(x, y) = \begin{cases} \frac{x^3 - y^3 + 2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) (8 marks) Find the partial derivatives $f_1(0, 0)$, $f_2(0, 0)$.

Solution: We have $f(x, 0) = \frac{x^3}{x^2} = x$ for $x \neq 0$, and $f(0, y) = -\frac{y^3}{y^2} = -y$ for $y \neq 0$. Since these formulas match $f(0, 0) = 0$, we have $f(x, 0) = x$ and $f(0, y) = -y$ for all x, y . Therefore $f_1(0, 0) = 1$ and $f_2(0, 0) = -1$.

- (b) (8 marks) If $f(x, y)$ differentiable at $(0, 0)$? Prove your answer.

Solution:

Let $x = y$, then $f(x, x) = \frac{2x^2}{2x^2} = 1$. As $x \rightarrow 0$, this does not approach $f(0, 0) = 0$. This means that f is not continuous at $(0, 0)$ and therefore not differentiable at $(0, 0)$.

10 marks

8. The quantities u, v are defined as functions of x, y so that they satisfy the equations $x = u \cos(u + v)$, $y = v \sin u$. Find $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$ at the point where $u = \pi/3$, $v = 0$. (Hint: $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, $\cos(\pi/3) = \frac{1}{2}$.)

Solution:

Differentiating the equations $x = u \cos(u + v)$, $y = v \sin u$ in x , we get

$$\begin{aligned}1 &= u_x \cos(u + v) - u \sin(u + v) \cdot (u_x + v_x), \\0 &= v_x \sin u + v \cos u \cdot u_x.\end{aligned}$$

At $u = \pi/3$, $v = 0$, we get

$$\begin{aligned}1 &= u_x \cos(\pi/3) - \frac{\pi}{3} \sin(\pi/3) \cdot (u_x + v_x) = \frac{u_x}{2} - \frac{\pi\sqrt{3}}{6}(u_x + v_x), \\0 &= v_x \sin(\pi/3) = \frac{\sqrt{3}}{2}v_x.\end{aligned}$$

Hence $v_x = 0$, and

$$u_x = \frac{1}{\frac{1}{2} - \frac{\pi\sqrt{3}}{6}} = \frac{6}{3 - \pi\sqrt{3}}.$$