*This midterm has **7 questions** on **4 pages**, for a total of 50 points.*

Duration: 50 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (including all middle names): SOLUTIONS

Student-No: ______________________

Signature: ______________________

Question:	1	2	3	4	5	6	7	Total
Points:	12	6	4	8	6	8	6	50
Score:								

12 marks 1. Decide whether each of the sets below is open, closed, or neither. What is the boundary and the interior of each set?

For this question only, an answer without explanation is sufficient.

(a) $\{(x, y, z) \in \mathbb{R}^3 : 0 < \sqrt{x^2 + y^2} \leq 3\}$

Solution: The set is neither open nor closed. The interior is the set $0 < \sqrt{x^2 + y^2} < 3$, and the boundary consists of the cylinder $\sqrt{x^2 + y^2} = 3$ and the line $x = y = 0$.

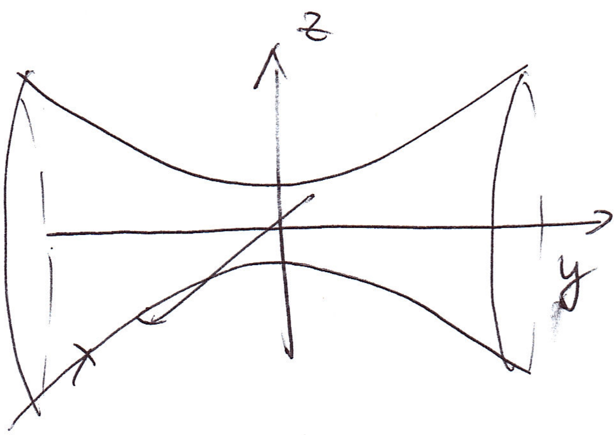
(b) $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0,\ x^2 + y^2 + z^2 \geq 4\}$

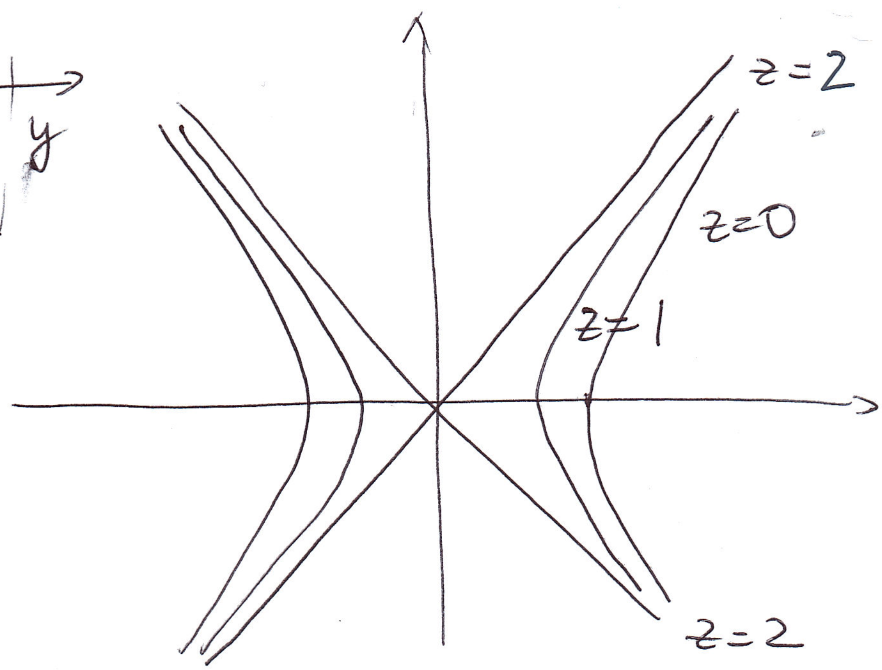
Solution: The set is closed. The interior is the set $x > 0,\ x^2 + y^2 + z^2 > 4$. The boundary consists of the half-sphere $x \geq 0,\ x^2 + y^2 + z^2 = 4$, and the set $x = 0,\ y^2 + z^2 \geq 4$.

6 marks 2. A surface in $\mathbb{R}^3$ has the equation $x^2 - y^2 + \frac{z^2}{4} = 1$. Sketch the surface. Find the equations of intersection of the surface with the planes $z = 0$, $z = 1$, $z = 2$, and sketch these curves.

Solution:

The surface is a hyperboloid of one sheet (pictured below). The intersection curves are the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = \frac{3}{4}$, and the pair of lines $x = \pm y$.





4 marks 3. A surface in $\mathbb{R}^3$ has the equation $R^2 - 4R\sin\phi + 3 = 0$ in spherical coordinates. Find its equation in cylindrical coordinates.

Solution:

We have $r = R\sin\theta$ and $R^2 = r^2 + z^2$, hence our equation in cylindrical coordinates is $r^2 + z^2 - 4r + 3 = 0$. (Optional: To see what kind of a surface this is, we rewrite the equation as $(r-2)^2 + z^2 = 1$. This is the torus obtained by rotating the circle $(x-2)^2 + z^3 = 1$ in the xz-plane around the z-axis.)

8 marks 4. Find the equations (in whichever form you prefer) of the line of intersection of the planes $3x - y + z = 2$ and $x + 2y - z = -4$.

Solution:

The two planes have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The direction vector $\mathbf{v}$ of the line should be perpendicular to them both, so we take

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}.$$

To find a point on the line, set e.g. $x = 0$ in the equations of both planes. We get $-y + z = 2$ and $2y - z = -4$. Adding these two equations we get $y = -2$, then from the first equation $z = y + 2 = 0$. Thus, $P = (0, 2, 0)$ is a point in the plane. (Of course there are other possibilities.)

Using P and $\mathbf{v}$ as above, the (vector parametric) equation of the line is

$$\mathbf{r} = \langle -t,\ -2 + 4t,\ 7t \rangle$$

6 marks 5. Find the volume of the parallelepiped in $\mathbb{R}^3$ spanned by the vectors $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $3\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + \mathbf{j}$.

Solution:

$$V = \left\| \begin{matrix} 1 & -1 & 4 \\ 0 & 3 & -1 \\ 2 & 1 & 0 \end{matrix} \right\| = |1 \cdot 1 + 1 \cdot 2 + 4 \cdot (-6)| = |-21| = 21$$

8 marks 6. Write the vector $\mathbf{w} = 3\mathbf{i} - 5\mathbf{j} + \sqrt{2}\mathbf{k}$ as a sum of two vectors $\mathbf{w} = \mathbf{u} + \mathbf{v}$ such that $\mathbf{u}$ is parallel to the plane $x + 2y - 2z = 0$ and the other is perpendicular to it.

Solution: The vector $\mathbf{v}$ should be the vector projection of $\mathbf{w}$ on a vector $\mathbf{n}$ normal to the plane. From the equation of the plane, we can take $\mathbf{n} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Then

$$\mathbf{v} = \mathbf{w}_{\mathbf{n}} = \frac{\mathbf{w} \cdot \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} = \frac{3 - 10 - 2\sqrt{2}}{1+4+4}\mathbf{n} = -\frac{7 + 2\sqrt{2}}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

$$= -\frac{7+2\sqrt{2}}{9}\mathbf{i} - \frac{14+4\sqrt{2}}{9}\mathbf{j} + \frac{14+4\sqrt{2}}{9}\mathbf{k}$$

and

$$\mathbf{u} = \mathbf{w} - \mathbf{v} = (3 + \frac{7+2\sqrt{2}}{9})\mathbf{i} + (-5 + \frac{14+4\sqrt{2}}{9})\mathbf{j} + (\sqrt{2} - \frac{14+4\sqrt{2}}{9})\mathbf{k}$$

$$= \frac{34+2\sqrt{2}}{9}\mathbf{i} + \frac{-31+4\sqrt{2}}{9}\mathbf{j} + \frac{-14+5\sqrt{2}}{9}\mathbf{k}$$

6 marks 7. Find the equation of the plane that contains the line $x = t, y = 2t, z = 0$, and the point $(3, -1, -1)$.

Solution: The plane has to be parallel to the vector $\mathbf{i}+2\mathbf{j}$ (the direction vector of the line). We are given that the point $(3, -1, -1)$ lies in the plane, and so does $(0, 0, 0)$ since it lies on the given line, so another vector parallel to the plane is $3\mathbf{i} - \mathbf{j} - \mathbf{k}$. We now find the normal vector

$$\mathbf{n} = (3\mathbf{i} - \mathbf{j} - \mathbf{k}) \times \mathbf{i} + 2\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -1 \\ 1 & 2 & 0 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} + 7\mathbf{k}.$$

The plane perpendicular to $\mathbf{n}$ and through $(0, 0, 0)$ is $2x - y + 7z = 0$.