1. Decide whether each of the sets below is open, closed, or neither. What is the boundary and the interior of each set?

For this question only, an answer without explanation is sufficient.

6 marks (a) $\{(x,y,z) \in \mathbb{R}^3 : \ z > \sqrt{x^2+y^2}, \ x^2+y^2+z^2 \le 2\}$

Solution:

- The set is neither open nor closed.
- The boundary is

$$\{(x,y,z) \in \mathbb{R}^3 : \ z = \sqrt{x^2+y^2}, \ x^2+y^2+z^2 \le 2\}$$

$$\cup\{(x,y,z) \in \mathbb{R}^3 : \ z > \sqrt{x^2+y^2}, \ x^2+y^2+z^2 = 2\}.$$

Optional: There are several ways to simplify the answer, for example

$$\{(x,y,z) \in \mathbb{R}^3 : \ z = \sqrt{x^2+y^2}, \ 0 \le z \le 1\}$$

$$\cup\{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2+z^2 = 2, \ z \ge 1\}.$$

- The interior is $\{(x,y,z) \in \mathbb{R}^3 : \ z > \sqrt{x^2+y^2}, \ x^2+y^2+z^2 < 2\}$.

6 marks (b) $\{(x,y,z) \in \mathbb{R}^3 : \ x > 0, \ 0 \le y \le 4\}$

Solution:

- The set is neither open nor closed.
- The boundary is

$$\{(x,y,z) \in \mathbb{R}^3 : \ x > 0, \ y = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 : \ x > 0. \ y = 4\}$$

$$\cup\{(x,y,z) \in \mathbb{R}^3 : \ x = 0, \ 0 \le y \le 4\}.$$

- The interior is $\{(x,y,z) \in \mathbb{R}^3 : \ x > 0, \ 0 < y < 4\}$.

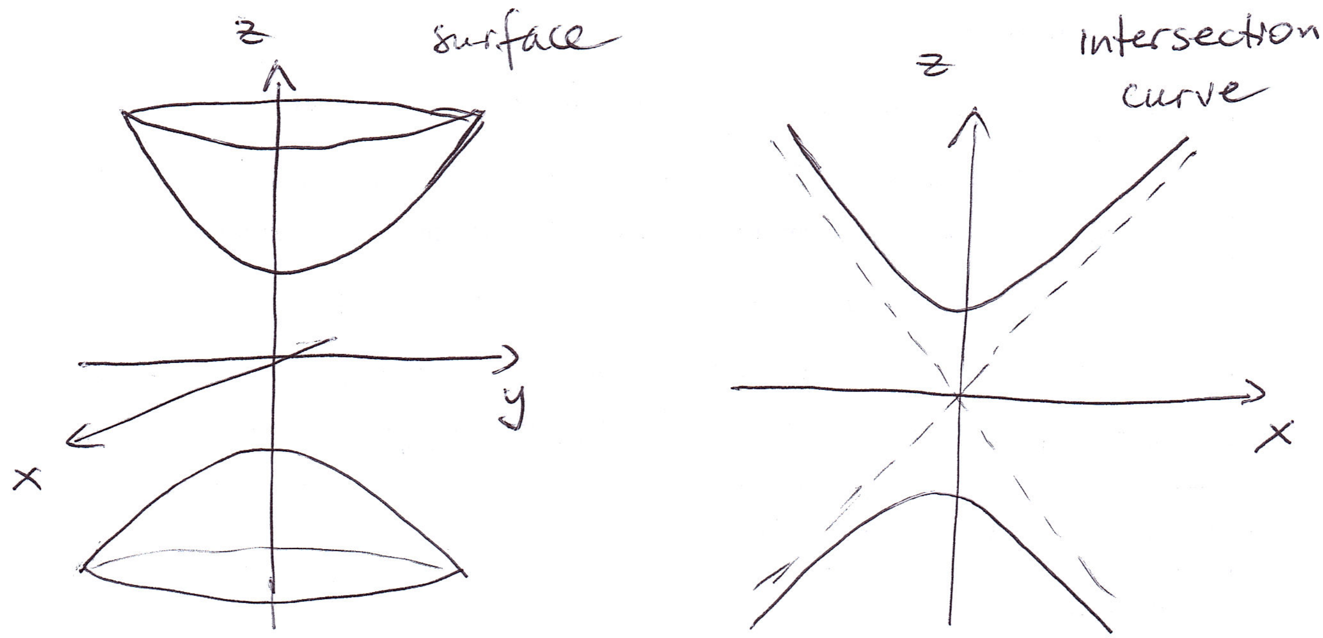
2. A surface in $\mathbb{R}^3$ has the equation $x^2 + y^2 = z^2 - 4$.

6 marks

(a) Sketch the surface. Find the equation of the intersection curve of the surface with the plane $y = 0$, and sketch this curve.

Solution:

The surface is a hyperboloid of two sheets around the z-axis. The intersection with the plane $y = 0$ is the hyperbola $z^2 = x^2 + 4$.



6 marks

(b) Convert the equation of this surface to cylindrical and spherical coordinates.

Solution:

- In cylindrical coordinates, we have $r^2 = x^2 + y^2$, so the equation of the surface is $r^2 = z^2 - 4$.
- In spherical coordinates, we have $r = R\sin\phi$ and $z = R\cos\phi$, so our equation is $R^2\sin^2\phi = R^2\cos^2\phi - 4$. (Optional: This can be simplified to $R^2(\cos^2\phi - \sin^2\phi) = 4$, or $R^2\cos(2\phi) = 4$.)

3. The points P, Q, R in $\mathbb{R}^3$ have coordinates $P = (1, 1, -1)$, $Q = (2, -1, 0)$, $R = (2, 3, 4)$.

4 marks (a) Find the area of the triangle $\triangle PQR$.

Solution: We have $\vec{PQ} = \langle 1, -2, 1\rangle$ and $\vec{PR} = \langle 1, 2, 5\rangle$, so that

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 1 & 2 & 5 \end{vmatrix} = -12\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} = -4(3\mathbf{i} + \mathbf{j} - \mathbf{k})$$

The area of the triangle is

$$\frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2} \cdot 4\sqrt{9+1+1} = 2\sqrt{11}.$$

4 marks (b) Find the angle at P in the triangle $\triangle PQR$. (An answer in the form $\cos^{-1}(\cdot)$ or $\sin^{-1}(\cdot)$ is sufficient.)

Solution: We have $|\vec{PQ}| = \sqrt{1+4+1} = \sqrt{6}$ and $|\vec{PR}| = \sqrt{1+4+25} = \sqrt{30}$. We also have $\vec{PQ} \cdot \vec{PR} = 1 - 4 + 5 = 2$. Therefore

$$\cos\theta = \frac{2}{\sqrt{6}\sqrt{30}} = \frac{1}{\sqrt{45}}, \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{45}}\right).$$

Alternative solution: it is also possible to find $\sin\theta$ from

$$\sin\theta = \frac{|\vec{PQ} \times \vec{PR}|}{|\vec{PQ}||\vec{PR}|} = \sqrt{\frac{44}{45}}.$$

However, this does not distinguish between the angles θ (in this case, acute) and $\pi - \theta$ (obtuse). The first method is preferred because it does provide that information.

4 marks (c) Find the equation (in the form $Ax + By + Cz = D$) of the plane through the points P, Q, R.

Solution:
The plane is perpendicular to the vector $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ from (a), and passes through $P = (1, 1, -1)$. Therefore the equation of the plane is

$$3(x-1) + (y-1) - (z+1) = 0,$$

which simplifies to $3x - 3 + y - 1 - z - 1 = 0$, or $3x + y - z = 5$.

2 marks 4. (a) Prove that the line $x = t+1, y = 3t, z = t+3$ is parallel to the plane $4x - y - z = 2$.

Solution: The line is parallel to the vector $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and the plane is perpendicular to the vector $\mathbf{v} = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$. SInce $\mathbf{u} \cdot \mathbf{v} = 4 - 3 - 1 = 0$, $\mathbf{u}$ is perpendicular to $\mathbf{v}$, therefore perpendicular to the plane.

6 marks (b) Find the equation (in the form $Ax + By + Cz = D$) of the plane that contains the line $x = t+1, y = 3t, z = t+3$ and is perpendicular to the plane $4x - y - z = 2$.

Solution:
The plane should be parallel to both $\mathbf{u}$ and $\mathbf{v}$ from (a), therefore perpendicular to their cross product:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 4 & -1 & -1 \end{vmatrix} = -2\mathbf{i} + 5\mathbf{j} - 13\mathbf{k}.$$

For a point that lies in the plane, we can take any point from the given line, for example with $t = 0$ we get $(1, 0, 3)$. Hence the equation of the plane is

$$-2(x-1) + 5(y-0) - 13(z-3) = 0,$$

which simplifies to $-2x - 2 + 3y - 13z + 39 = 0$, or $2x - 3y + 13z = 41$.

6 marks (c) Let $P = (2, 5, 1)$. Find the point Q on the line $x = t+1, y = 3t, z = t+3$ which is closest to P.

Solution: Let $Q = (t+1, 3t, t+3)$ be a point on the line, for some t that we need to find. For Q to be the closest point to P, the vector $\vec{PQ} = \langle t+1-2, 3t-5, t+3-1\rangle = \langle t-1, 3t-5, t+2\rangle$ should be perpendicular to the line, hence to $\mathbf{u}$. We set up the equation for that:

$$\vec{PQ} \cdot \mathbf{u} = t - 1 + (3t-5) \cdot 3 + (t+2) = 0,$$

which simplifies to

$$t - 1 + 9t - 15 + t + 2 = 0, \quad 11t = 14, \quad t = \frac{14}{11}.$$

Therefore Q has coordinates

$$x = \frac{14}{11} + 1 = \frac{25}{11}, \quad y = 3 \cdot \frac{14}{11} = \frac{42}{11}, \quad z = \frac{14}{11} + 3 = \frac{47}{11}.$$