*This midterm has **5 questions** on **5 pages**, for a total of 50 points.*

Duration: 50 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Last name: ______________________________ Student No.: ______________

First name and all middle names: ______________________________

Signature: ______________________________

Question:	1	2	3	4	5	Total
Points:	12	8	8	12	10	50
Score:						

12 marks 1. Let

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (8 marks) Does f have a limit at $(0,0)$? Explain your answer.

Solution:
Since $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$, we have for all $(x,y) \neq (0,0)$

$$\left|\frac{xy^3}{x^2+y^2}\right| \leq \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2.$$

Since $x^2+y^2 \to 0$ as $(x,y) \to (0,0)$, the function has limit 0 at that point. (In terms of δ and ϵ, to ensure that $|f(x,y)-0| < \delta$, it suffices if $x^2+y^2 < \delta$, or equivalently, $\sqrt{x^2+y^2} < \sqrt{\delta}$. Therefore we can take $\epsilon = \sqrt{\delta}$.)

(b) (4 marks) Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$, or else explain why they do not exist.

Solution: We have $f(x,0) = 0$ for all x and $f(0,y) = 0$ for all y, so that $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

8 marks 2. Find the equation of the tangent plane to the surface $z = 5x^2y - y^2$ at the point where $x = -1$, $y = 3$.

Solution:

We have $\dfrac{\partial z}{\partial x} = 10xy$ and $\dfrac{\partial z}{\partial y} = 5x^2 - 2y$. Thus at $(x, y) = (-1, 3)$, $z = 15 - 9 = 6$, $\dfrac{\partial z}{\partial x} = -30$, and $\dfrac{\partial z}{\partial y} = 5 - 6 = -1$. Thus the equation of the tangent plane is

$$z = 6 - 30(x + 1) - (y - 3)$$

which simplifies to $z = -30x - y - 21$.

8 marks 3. Let $w = f(a_1x + a_2y + a_3z, b_1x + b_2y + b_3z)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous first order partial derivatives. Prove that

$$c_1\frac{\partial w}{\partial x} + c_2\frac{\partial w}{\partial y} + c_3\frac{\partial w}{\partial z} = 0$$

for any vector (c_1, c_2, c_3) orthogonal to both (a_1, a_2, a_3) and (b_1, b_2, b_3).

Solution:

We have $w = f(u, v)$, where $u = a_1x + a_2y + a_3z$ and $v = b_1x + b_2y + b_3z$. By the Chain Rule,

$$c_1\frac{\partial w}{\partial x} + c_2\frac{\partial w}{\partial y} + c_3\frac{\partial w}{\partial z} = c_1(f_ua_1 + f_vb_1) + c_2(f_ua_2 + f_vb_2) + c_3(f_ua_3 + f_vb_3)$$

$$= (c_1a_1 + c_2a_2 + c_3a_3)f_u + (c_1b_1 + c_2b_2 + c_3b_3)f_v = (\mathbf{c} \cdot \mathbf{a})f_u + (\mathbf{c} \cdot \mathbf{b})f_v = 0.$$

12 marks

4. The tangent plane to the graph of $z = f(x, y)$ at $(x, y) = (3, 4)$ has the equation $x - 6y - 2z = 1$.

(a) Find the directional derivative $D_{\mathbf{u}} f(3, 4)$ if $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$.

Solution:
From the equation of the tangent plane, we have $\nabla f(3, 4) = \langle 1/2, -3 \rangle$ so that

$$D_{\mathbf{u}} f(3, 4) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - 3(-\frac{1}{2}) = \frac{\sqrt{3}}{4} + \frac{3}{2}.$$

(b) Is there a unit vector $\mathbf{v}$ such that $D_{\mathbf{v}} f(3, 4) = 4$? If yes, find it. If no, explain why.

Solution: (3 marks)
The largest possible value of $D_{\mathbf{v}} f(3, 4)$ is $|\nabla f(3, 4)| = \sqrt{(1/2)^2 + 3^2} = \sqrt{9.25}$. This is less than 4 (since $9.25 < 16$). Hence there is no such $\mathbf{v}$.

10 marks 5. The equations

$$u = x^3 - y^2$$
$$v = 2xy^3$$

define x, y implicitly as functions of u, v near $x = -1$, $y = 1$. Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ at $x = -1$, $y = 1$.

Solution:

Differentiating the two equations with respect to u, we get

$$1 = 3x^2 \frac{\partial x}{\partial u} - 2y \frac{\partial y}{\partial u}$$
$$0 = 2y^3 \frac{\partial x}{\partial u} + 6xy^2 \frac{\partial y}{\partial u}$$

so that

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix} 1 & -2y \\ 0 & 6xy^2 \end{vmatrix}}{\begin{vmatrix} 3x^2 & -2y \\ 2y^3 & 6xy^2 \end{vmatrix}} = \frac{6xy^2}{18x^3y^2 + 4y^4} = \frac{3x}{9x^3 + 2y^2}$$

$$\frac{\partial y}{\partial u} = \frac{\begin{vmatrix} 3x^2 & 1 \\ 2y^3 & 0 \end{vmatrix}}{\begin{vmatrix} 3x^2 & -2y \\ 2y^3 & 6xy^2 \end{vmatrix}} = \frac{-2y^3}{18x^3y^2 + 4y^4} = \frac{-y}{9x^3 + 2y^2}$$

Thus at $(x, y) = (-1, 1)$, we have

$$\frac{\partial x}{\partial u} = \frac{-3}{-9+2} = \frac{3}{7}, \quad \frac{\partial y}{\partial u} = \frac{-1}{-9+2} = -\frac{1}{7}.$$