*This midterm has **4 questions** on **5 pages**, for a total of 50 points.*

Duration: 50 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Last name: ______________________________ Student No.: ______________

First name and all middle names: ______________________________

Signature: ______________________________

Question:	1	2	3	4	Total
Points:	16	10	14	10	50
Score:					

16 marks

1. Let $f(x,y) = x\sin(xy^2)$.

(a) (8 marks) Find the equation of the tangent plane to the surface $z = f(x,y)$ at the point where $x = \pi$, $y = 1$.

Solution:
We have $f_1(x,y) = \sin(xy^2) + x\cos(xy^2)\cdot y^2$ and $f_2(x,y) = x\cos(xy^2)\cdot 2xy$, so that $f_1(\pi,1) = \sin\pi + \pi\cos\pi\cdot 1 = 0 - \pi = -\pi$ and $f_2(\pi,1) = \pi\cos\pi\cdot 2\pi = -2\pi^2$. We also have $f(\pi,1) = \pi\sin\pi = 0$. Therefore the tangent plane is

$$z = -\pi(x-\pi) - 2\pi^2(y-1) = -\pi x - 2\pi^2 y + 3\pi^2.$$

(b) (4 marks) Find the directional derivative of f at $(\pi,1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

Solution: The unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{i}$. From (a), we have $\nabla f(\pi,1) = -\pi\mathbf{i} - 2\pi^2\mathbf{j}$. Therefore

$$D_{\mathbf{u}}f(\pi,1) = \nabla f(\pi,1)\cdot\mathbf{u} = \frac{1}{\sqrt{10}}(-\pi) - \frac{3}{\sqrt{10}}(-2\pi^2) = \frac{\pi}{\sqrt{10}}(6\pi - 1)$$

(c) (4 marks) Find a vector $\mathbf{w}$ in the xy-plane tangent to the level curve of f at $(\pi,1)$. (The answer might not be unique. Any non-zero tangent vector will suffice.)

Solution:
The tangent vector should be perpendicular to $\nabla f(\pi,1) = -\pi\mathbf{i} - 2\pi^2\mathbf{j}$. For example, we could take $\mathbf{w} = 2\pi^2\mathbf{i} - \pi\mathbf{j}$.

10 marks

2. Suppose that a function $f(x, y)$ is defined everywhere in the plane and has continuous partial derivatives of all orders. Assume that $f(2, 3) = 2$, $f_1(2, 3) = -1$, $f_2(2, 3) = 4$, $f_{11}(2, 3) = 0$, $f_{12}(2, 3) = f_{21}(2, 3) = 3$, $f_{22}(2, 3) = 1$, Find $\frac{\partial}{\partial x} f(2x^2, x + y)$ and $\frac{\partial^2}{\partial x^2} f(2x^2, x + y)$ at $x = 1$, $y = 2$.

Solution:

We start with

$$\frac{\partial}{\partial x} f(2x^2, x + y) = f_1(2x^2, x + y) \cdot (4x) + f_2(2x^2, x + y) \cdot 1.$$

At $x = 1, y = 2$, we have $2x^2 = 2$, $x + y = 3$, and

$$\frac{\partial}{\partial x} f(2x^2, x + y)\Big|_{(x,y)=(1,2)} = f_1(2, 3) \cdot 4 + f_2(2, 3) \cdot 1 = -1 \cdot 4 + 4 \cdot 1 = 0.$$

Next, using Chain Rule again,

$$\begin{aligned}\frac{\partial^2}{\partial x^2} f(2x^2, x + y) &= \frac{\partial}{\partial x}\Big(f_1(2x^2, x + y) \cdot (4x)\Big) + \frac{\partial}{\partial x} f_2(2x^2, x + y) \\ &= \Big(f_{11}(2x^2, x + y) \cdot (4x) + f_{21}(2x^2, x + y) \cdot 1\Big)(4x) + f_1(2x^2, x + y) \cdot 4 \\ &\quad + f_{21}(2x^2, x + y) \cdot (4x) + f_{22}(2x^2, x + y) \cdot 1.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial^2}{\partial x^2} f(2x^2, x + y)\Big|_{(x,y)=(1,2)} &= \Big(f_{11}(2, 3) \cdot 4 + f_{21}(2, 3)\Big) \cdot 4 \\ &\quad + f_1(2, 3) \cdot 4 + f_{21}(2, 3) \cdot 4 + f_{22}(2, 3) \\ &= (0 \cdot 4 + 3) \cdot 4 + (-1) \cdot 4 + 3 \cdot 4 + 1 \\ &= 12 - 4 + 12 + 1 = 21.\end{aligned}$$

14 marks 3. Let

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (6 marks) Is f continuous at $(0,0)$? Prove your answer.

Solution:
For all $(x,y) \neq (0,0)$, we have

$$|f(x,y| = \left|\frac{x^3}{x^2+y^2}\right| = |x| \cdot \left|\frac{x^2}{x^2+y^2}\right| \leq |x|.$$

Since $|x| \to 0$ as $(x,y) \to (0,0)$, we also have $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. Therefore f is continuous at $(0,0)$.

(b) (8 marks) Is it differentiable at $(0,0)$? Prove your answer.

Solution:
We have $f(x,0) = x$ for all $x \in \mathbb{R}$ and $f(0,y) = 0$ for all $y \in \mathbb{R}$, so that $f_1(0,0) = 1$ and $f_2(0,0) = 0$. Therefore the linear approximation to $f(x,y)$ at $(0,0)$, if it exists, is $L(x,y) = f(0,0) + f_1(0,0)x + f_2(0,0)y = x$.
For f to be differentiable at $(0,0)$, we need

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = 0. \tag{1}$$

We have

$$f(x,y) - L(x,y) = \frac{x^3}{x^2+y^2} - x = \frac{x^3 - x^3 - xy^2}{x^2+y^2} = \frac{xy^2}{x^2+y^2}.$$

Therefore the limit in (1) is

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}}.$$

But this limit is not 0 (in fact, the limit does not exist). For example, if we let $x > 0$, $y = x$, and then take $x \to 0$, then we get

$$\lim_{x\to 0} \frac{x^3}{(2x^2)^{3/2}} = \lim_{x\to 0} \frac{x^3}{2^{3/2}x^3} = \lim_{x\to 0} \frac{1}{2^{3/2}} = 2^{-3/2} \neq 0.$$

10 marks 4. A surface is given by the equation $e^x(y+1) - e^y z = (e^z - 1)x$.

(a) (2 marks) Prove that the point $P = (1, 0, 1)$ lies on the surface.

Solution: We just check that $(1, 0, 1)$ satisfy the equation of the surface: $e^1(0+1) - e^0 \cdot 1 = (e^1 - 1) \cdot 1$, $e - 1 = e - 1$, which is true.

(b) (8 marks) Explain why this equation defines z as a function of x and y in a neighbourhood of P. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at P.

Solution:

We can rewrite the equation of the surface as $F(x, y, z) = 0$, where $F(x, y, z) = e^x(y+1) - e^y z - (e^z - 1)x$. Then F has continuous partial derivatives of all orders, and

$$\begin{aligned} F_x(x,y,z) &= e^x(y+1) - (e^z - 1) \\ F_y(x,y,z) &= e^x - e^y z \\ F_z(x,y,z) &= -e^y - e^z x \end{aligned}$$

So

$$\begin{aligned} F_x(1,0,1) &= e^1(0+1) - (e^1 - 1) = 1 \\ F_y(1,0,1) &= e^1 - e^0 \cdot 1 = e - 1 \\ F_z(1,0,1) &= -e^0 - e^1 \cdot 1 = -1 - e \end{aligned}$$

Since $F_z(1, 0, 1) \neq 0$, the equation defines z as a function of x, y in a neighbourhood of P, and we have at P

$$\frac{\partial z}{\partial x} = -\frac{F_x(1,0,1)}{F_z(1,0,1)} = \frac{1}{e+1}, \quad \frac{\partial z}{\partial y} = -\frac{F_y(1,0,1)}{F_z(1,0,1)} = \frac{e-1}{e+1}.$$