

*This final exam has **7 questions** on **9 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Last name: _____ First name: _____

Student no.: _____ Course Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	Total
Points:	16	12	12	12	12	20	16	100
Score:								

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Please read the following points carefully before starting to write.

- In all questions except Q2, give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Write clearly and legibly, in complete sentences. Make sure that the logic of your argument is clear.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- If you run out of space, continue on the back of the page, on the blank page at the end, or in an additional booklet, **with clear indication on the original page of where the solution is continued.**

16 marks

1. Give examples of the following. Justify your answers.

(a) (4 marks) Three non-zero vectors \mathbf{u} , \mathbf{v} , \mathbf{w} , such that $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$ but $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

Solution: $\mathbf{u} = \mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$ is one example. Then $\mathbf{v} \times \mathbf{w} = \mathbf{i} \times \mathbf{j} = \mathbf{k} \neq \mathbf{0}$ and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{i} \cdot \mathbf{k} = 0$. Any other non-zero vectors such that \mathbf{v} and \mathbf{w} are not colinear but \mathbf{u} , \mathbf{v} , \mathbf{w} are coplanar would also work.

(b) (6 marks) A line parallel to the plane $x + y + z = 0$ but not contained in that plane. Give the equation of the line in the vector parametric form $\mathbf{r} = \mathbf{a} + t\mathbf{v}$.

Solution: The plane is perpendicular to the vector $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, so \mathbf{v} should also be perpendicular to it. Additionally, \mathbf{a} should not satisfy the equation of the plane. Thus we need $v_1 + v_2 + v_3 = 0$ and $a_1 + a_2 + a_3 \neq 0$, e.g. $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{a} = \mathbf{i}$.

(c) (6 marks) Two different functions $f(x, y)$ and $g(x, y)$, defined and continuous for all x, y , whose level curves are parabolas $y = x^2 + C$ for $C \in \mathbb{R}$.

Solution: We can take for instance $f(x, y) = y - x^2$ and $g(x, y) = y - x^2 + 1$. Then the level curves $f(x, y) = C$ and $g(x, y) = C + 1$ have the form we need.

12 marks

2. Decide whether each statement is True or False. In this question only, you do not need to justify your answers.

- (a) (3 marks) For every function $f(x, y)$ defined on an open set containing (a, b) , if the first order partial derivatives $f_1(a, b)$ and $f_2(a, b)$ exist, then f is differentiable at (a, b) .

Solution: False.

- (b) (3 marks) For every function $f(x, y)$ defined on an open set containing (a, b) , if the first order partial derivatives $f_1(a, b)$ and $f_2(a, b)$ exist, then f is continuous at (a, b) .

Solution: False.

- (c) (3 marks) For every function $f(x, y)$ defined on an open set containing (a, b) , if the first order partial derivatives $f_1(x, y)$ and $f_2(x, y)$ exist and are continuous in some neighbourhood of (a, b) , then f is differentiable at (a, b) .

Solution: True.

- (d) (3 marks) For every function $f(x, y)$ defined on an open set containing (a, b) , if f is differentiable at (a, b) , it is continuous at (a, b) .

Solution: True.

12 marks

3. Find all points $P = (1, b, c)$ such that P lies on the paraboloid $z = x^2 + y^2$ and the tangent plane to this paraboloid at P passes through the point $(2, 4, 18)$.

Solution:

The gradient of $f(x, y) = x^2 + y^2$ is $\nabla f(x, y) = \langle 2a, 2b \rangle$, so the equation of the tangent plane at (a, b, c) is $z - c = 2a(x - a) + 2b(y - b) = 2ax + 2by - 2a^2 - 2b^2$. Letting $a = 1$, and using also that $c = a^2 + b^2 = 1 + b^2$, we get the equation

$$z = 2x + 2by - 1 - b^2.$$

The point $(2, 4, 18)$ should satisfy that equation: $18 = 4 + 8b - 1 - b^2$, $b^2 - 8b + 15 = 0$, $b = 3$ or 5 . The corresponding points on the paraboloid are $(1, 3, 10)$ and $(1, 5, 26)$.

12 marks

4. Let $f(x, y) = x^4 - 4xy + y^2$. Find all critical points of f and classify them as local minima, maxima, or saddle points.

Solution: We have

$$f_1(x, y) = 4x^3 - 4y, \quad f_2(x, y) = -4x + 2y.$$

For $f_1 = f_2 = 0$, we need $y = 2x$ and $4x^3 = 4y = 8x$, $x^3 = 2x$, so that $x = 0$ or $x = \pm\sqrt{2}$. This corresponds to three critical points $(0, 0)$, $(\sqrt{2}, 2\sqrt{2})$, $(-\sqrt{2}, -2\sqrt{2})$.

The second order partials are $f_{11}(x, y) = 12x^2$, $f_{12}(x, y) = f_{21}(x, y) = -4$, $f_{22}(x, y) = 2$.

- At $(0, 0)$,

$$\begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} = -16 < 0,$$

so by the second derivative test, f has a saddle point.

- At $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$,

$$\begin{vmatrix} 24 & -4 \\ -4 & 2 \end{vmatrix} = 48 - 16 > 0, \quad f_{11} = 24 > 0$$

so by the second derivative test, f has local minima at these points.

12 marks

5. Find the largest and smallest values of the function $f(x, y, z) = x + 2y$ in the closed disc $x^2 + y^2 \leq 25$.

Solution:

Since $\nabla f = \langle 1, 2 \rangle \neq 0$, there are no critical points. Therefore the largest and smallest values are attained on the circle $x^2 + y^2 = 25$. We use Lagrange multipliers with $g(x, y) = x^2 + y^2 - 25$: $\nabla g = \langle 2x, 2y \rangle$, so that for $\nabla f = \lambda \nabla g$ we must have $1 = 2\lambda x$, $2 = 2\lambda y$. Hence $\lambda \neq 0$ and $2\lambda y = 2 = 4\lambda x$, $y = 2x$. Plugging this into the equation of the circle we get $x^2 + (2x)^2 = 5x^2 = 25$, $x^2 = 5$, $x = \pm\sqrt{5}$.

This yields two critical points, $(\sqrt{5}, 2\sqrt{5})$ and $(-\sqrt{5}, -2\sqrt{5})$. We have $f(\sqrt{5}, 2\sqrt{5}) = 5\sqrt{5}$, the largest value, and $f(-\sqrt{5}, -2\sqrt{5}) = -5\sqrt{5}$, the smallest value.

20 marks

6. Evaluate the double integrals.

(a) (10 marks) $\iint_D y \, dA$, where $D = \{(x, y) : y \geq 0, 1 \leq x^2 + y^2 \leq 9\}$.

Solution:

$$\begin{aligned} \int_0^\pi \int_1^3 r \sin \theta \, r \, dr \, d\theta &= \int_0^\pi \int_1^3 r^2 \sin \theta \, dr \, d\theta \\ &= \int_0^\pi \frac{1}{3} r^3 \sin \theta \Big|_{r=1}^3 \, d\theta = \int_0^\pi \left(9 - \frac{1}{3}\right) \sin \theta \, d\theta \\ &= \int_0^\pi \frac{26}{3} (-\cos \theta) \Big|_0^\pi = \frac{26}{3} \cdot 2 = \frac{52}{3} \end{aligned}$$

(b) (10 marks) $\int_0^\pi \int_{2x}^{2\pi} \cos(y^2) \, dy \, dx$. (Hint: change the order of integration.)

Solution:

$$\begin{aligned} \int_0^{2\pi} \int_0^{y/2} \cos(y^2) \, dx \, dy &= \int_0^{2\pi} \frac{y}{2} \cos(y^2) \, dy \\ &= \frac{1}{4} \sin(y^2) \Big|_0^{2\pi} = \frac{\sin(4\pi^2)}{4} \end{aligned}$$

16 marks

7. Let

$$f(x, y) = \begin{cases} \frac{3x^4 - y^3}{x^4 + 2y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) (8 marks) Is f continuous at $(0, 0)$? Prove your answer.

Solution: Since $f(x, 0) = 3 \rightarrow 3$ as $x \rightarrow 0$, $x \neq 0$, and $f(0, 0) = 0$, f is not continuous at $(0, 0)$. (In fact, f does not have a limit at this point.)

(b) (8 marks) Find the directional derivative $D_{\mathbf{u}}f(0, 0)$ for $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$, or explain why it does not exist.

Solution:

$$\begin{aligned} D_{\mathbf{u}}f(0, 0) &= \lim_{t \rightarrow 0} \frac{1}{t} \left(f\left(\frac{\sqrt{3}t}{2}, \frac{t}{2}\right) - f(0, 0) \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \frac{3\left(\frac{\sqrt{3}t}{2}\right)^4 - \left(\frac{t}{2}\right)^3}{\left(\frac{\sqrt{3}t}{2}\right)^4 + 2\left(\frac{t}{2}\right)^2} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \frac{\frac{27}{16}t^4 - \frac{1}{8}t^3}{\frac{9}{16}t^4 + \frac{1}{2}t^2} \\ &= \lim_{t \rightarrow 0} \frac{\frac{27}{16}t^4 - \frac{1}{8}t^3}{\frac{9}{16}t^5 + \frac{1}{2}t^3} \\ &= \lim_{t \rightarrow 0} \frac{\frac{27}{16}t - \frac{1}{8}}{\frac{9}{16}t^2 + \frac{1}{2}} = \frac{-1/8}{1/2} = -\frac{1}{4}. \end{aligned}$$