*This final exam has **7 questions** on **9 pages**, for a total of 100 marks.*
Duration: 2 hours 30 minutes

Last name: ______________________ First name: ______________________

Student no.: ______________________ Course Section: ______________________

Signature: ____

Question:	1	2	3	4	5	6	7	Total
Points:	16	12	12	12	12	20	16	100
Score:								

Please read the following points carefully before starting to write.

- In all questions except Q2, give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Write clearly and legibly, in complete sentences. Make sure that the logic of your argument is clear.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- If you run out of space, continue on the back of the page, on the blank page at the end, or in an additional booklet, **with clear indication on the original page of where the solution is continued**.

16 marks

1. Give examples of the following. Justify your answers.

(a) (4 marks) Three non-zero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, such that $\mathbf{v} \times \mathbf{w} \neq 0$ but $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

Solution: $\mathbf{u} = \mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$ is one example. Then $\mathbf{v} \times \mathbf{w} = \mathbf{i} \times \mathbf{j} = \mathbf{k} \neq 0$ and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{i} \cdot \mathbf{k} = 0$. Any other non-zero vectors such that $\mathbf{v}$ and $\mathbf{w}$ are not colinear but $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are coplanar would also work.

(b) (6 marks) A line parallel to the plane $x + y + z = 0$ but not contained in that plane. Give the equation of the line in the vector parametric form $\mathbf{r} = \mathbf{a} + t\mathbf{v}$.

Solution: The plane is perpendicular to the vector $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, so $\mathbf{v}$ should also be perpendicular to it. Additionally, $\mathbf{a}$ should not satisfy the equation of the plane. Thus we need $v_1 + v_2 + v_3 = 0$ and $a_1 + a_2 + a_3 \neq 0$, e.g. $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{a} = \mathbf{i}$.

(c) (6 marks) Two different functions $f(x, y)$ and $g(x, y)$, defined and continuous for all x, y, whose level curves are parabolas $y = x^2 + C$ for $C \in \mathbb{R}$.

Solution: We can take for instance $f(x, y) = y - x^2$ and $g(x, y) = y - x^2 + 1$. Then the level curves $f(x, y) = C$ and $g(x, y) = C + 1$ have the form we need.

12 marks 2. Decide whether each statement is True or False. In this question only, you do not need to justify your answers.

(a) (3 marks) For every function $f(x, y)$ defined on an open set containing (a, b), if the first order partial derivatives $f_1(a, b)$ and $f_2(a, b)$ exist, then f is differentiable at (a, b).

Solution: False.

(b) (3 marks) For every function $f(x, y)$ defined on an open set containing (a, b), if the first order partial derivatives $f_1(a, b)$ and $f_2(a, b)$ exist, then f is continuous at (a, b).

Solution: False.

(c) (3 marks) For every function $f(x, y)$ defined on an open set containing (a, b), if the first order partial derivatives $f_1(x, y)$ and $f_2(x, y)$ exist and are continuous in some neighbourhood of (a, b), then f is differentiable at (a, b).

Solution: True.

(d) (3 marks) For every function $f(x, y)$ defined on an open set containing (a, b), if f is differentiable at (a, b), it is continuous at (a, b).

Solution: True.

12 marks 3. Find all points $P = (1, b, c)$ such that P lies on the paraboloid $z = x^2 + y^2$ and the tangent plane to this paraboloid at P passes through the point $(2, 4, 18)$.

Solution:

The gradient of $f(x, y) = x^2+y^2$ is $\nabla f(x, y) = \langle 2a, 2b\rangle$, so the equation of the tangent plane at (a, b, c) is $z - c = 2a(x - a) + 2b(y - b) = 2ax + 2by - 2a^2 - 2b^2$. Letting $a = 1$, and using also that $c = a^2 + b^2 = 1 + b^2$, we get the equation

$$z = 2x + 2by - 1 - b^2.$$

The point $(2, 4, 18)$ should satisfy that equation: $18 = 4+8b-1-b^2$, $b^2-8b+15 = 0$, $b = 3$ or 5. The corresponding points on the paraboloid are $(1, 3, 10)$ and $(1, 5, 26)$.

12 marks 4. Let $f(x, y) = x^4 - 4xy + y^2$. Find all critical points of f and classify them as local minima, maxima, or saddle points.

Solution: We have

$$f_1(x, y) = 4x^3 - 4y, \quad f_2(x, y) = -4x + 2y.$$

For $f_1 = f_2 = 0$, we need $y = 2x$ and $4x^3 = 4y = 8x$, $x^3 = 2x$, so that $x = 0$ or $x = \pm\sqrt{2}$. This corresponds to three critical points $(0, 0)$, $(\sqrt{2}, 2\sqrt{2})$, $(-\sqrt{2}, -2\sqrt{2})$. The second order partials are $f_{11}(x, y) = 12x^2$, $f_{12}(x, y) = f_{21}(x, y) = -4$, $f_{22}(x, y) = 2$.

- At $(0, 0)$,
$$\begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} = -16 < 0,$$
so by the second derivative test, f has a saddle point.
- At $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$,
$$\begin{vmatrix} 24 & -4 \\ -4 & 2 \end{vmatrix} = 48 - 16 > 0, \quad f_{11} = 24 > 0$$
so by the second derivative test, f has local minima at these points.

12 marks 5. Find the largest and smallest values of the function $f(x, y, z) = x + 2y$ in the closed disc $x^2 + y^2 \leq 25$.

Solution:

Since $\nabla f = \langle 1, 2\rangle \neq 0$, there are no critical points. Therefore the largest and smallest values are attained on the circle $x^2 + y^2 = 25$. We use Lagrange multipliers with $g(x, y) = x^2 + y^2 - 25$: $\nabla g = \langle 2x, 2y\rangle$, so that for $\nabla f = \lambda \nabla g$ we must have $1 = 2\lambda x$, $2 = 2\lambda y$. Hence $\lambda \neq 0$ and $2\lambda y = 2 = 4\lambda x$, $y = 2x$. Plugging this into the equation of the circle we get $x^2 + (2x)^2 = 5x^2 = 25$, $x^2 = 5$, $x = \pm\sqrt{5}$.

This yields two critical points, $(\sqrt{5}, 2\sqrt{5})$ and $(-\sqrt{5}, -2\sqrt{5})$. We have $f(\sqrt{5}, 2\sqrt{5}) = 5\sqrt{5}$, the largest value, and $f(-\sqrt{5}, -2\sqrt{5}) = -5\sqrt{5}$, the smallest value.

20 marks 6. Evaluate the double integrals.

(a) (10 marks) $\iint_D y\,dA$, where $D = \{(x,y) :\ y \geq 0,\ 1 \leq x^2+y^2 \leq 9\}$.

Solution:

$$\int_0^{\pi}\int_1^3 r\sin\theta\, r dr\, d\theta = \int_0^{\pi}\int_1^3 r^2\sin\theta\, dr\, d\theta$$
$$= \int_0^{\pi} \frac{1}{3}r^3\sin\theta\Big|_{r=1}^{3} d\theta = \int_0^{\pi}(9-\frac{1}{3})\sin\theta\, d\theta$$
$$= \int_0^{\pi}\frac{26}{3}(-\cos\theta)\Big|_0^{\pi} = \frac{26}{3}\cdot 2 = \frac{52}{3}$$

(b) (10 marks) $\int_0^{\pi}\int_{2x}^{2\pi}\cos(y^2)dy\,dx$. (Hint: change the order of integration.)

Solution:

$$\int_0^{2\pi}\int_0^{y/2}\cos(y^2)dx\,dy = \int_0^{2\pi}\frac{y}{2}\cos(y^2)dy$$
$$= \frac{1}{4}\sin(y^2)\Big|_0^{2\pi} = \frac{\sin(4\pi^2)}{4}$$

16 marks

7. Let

$$f(x,y) = \begin{cases} \dfrac{3x^4 - y^3}{x^4 + 2y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (8 marks) Is f continuous at $(0,0)$? Prove your answer.

Solution: Since $f(x,0) = 3 \to 3$ as $x \to 0$, $x \neq 0$, and $f(0,0) = 0$, f is not continuous at $(0,0)$. (In fact, f does not have a limit at this point.)

(b) (8 marks) Find the directional derivative $D_{\mathbf{u}} f(0,0)$ for $\mathbf{u} = \dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$, or explain why it does not exist.

Solution:

$$\begin{aligned}
D_{\mathbf{u}} f(0,0) &= \lim_{t\to 0} \frac{1}{t}\left(f\left(\frac{\sqrt{3}t}{2}, \frac{t}{2} \right) - f(0,0) \right) \\
&= \lim_{t\to 0} \frac{1}{t} \frac{3\left(\frac{\sqrt{3}t}{2}\right)^4 - \left(\frac{t}{2}\right)^3}{\left(\frac{\sqrt{3}t}{2}\right)^4 + 2\left(\frac{t}{2}\right)^2} \\
&= \lim_{t\to 0} \frac{1}{t} \frac{\frac{27}{16}t^4 - \frac{1}{8}t^3}{\frac{9}{16}t^4 + \frac{1}{2}t^2} \\
&= \lim_{t\to 0} \frac{\frac{27}{16}t^4 - \frac{1}{8}t^3}{\frac{9}{16}t^5 + \frac{1}{2}t^3} \\
&= \lim_{t\to 0} \frac{\frac{27}{16}t - \frac{1}{8}}{\frac{9}{16}t^2 + \frac{1}{2}} = \frac{-1/8}{1/2} = -\frac{1}{4}.
\end{aligned}$$