This final exam has ***9 questions*** *on* ***11 pages****, for a total of 100 marks.*
Duration: 2 hours 30 minutes

Last name: ____________________ First name: ____________________

Student no.: ____

Signature: ____

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	12	12	12	12	12	10	8	10	12	100
Score:										

Please read the following points carefully before starting to write.

- In all questions except Q1 and 2, give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Write clearly and legibly, in complete sentences. Make sure that the logic of your argument is clear.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- If you run out of space, continue on the back of the page, on the blank page at the end, or in an additional booklet, **with clear indication on the original page of where the solution is continued**.

12 marks

1. Give examples of the following. (In this question, you do not need to justify your answers.)

   (a) (3 marks) Three non-zero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ such that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4$.

   Solution: $\mathbf{u} = 4\mathbf{k}$, $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$ is one example. Then $\mathbf{v} \times \mathbf{w} = \mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4$.

   (b) (3 marks) Three non-zero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ such that $\mathbf{u} \neq \mathbf{v}$ but $\mathbf{u}$ and $\mathbf{v}$ have the same vector projections on $\mathbf{w}$.

   Solution: Let $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$, $\mathbf{w} = \mathbf{k}$, then both vector projections are $\mathbf{0}$.

   (c) (3 marks) A function f defined on $\mathbb{R}^2$ which has a global minimum at $(2, 5)$ and no other local or global maxima or minima.

   Solution:
   $f(x, y) = (x - 2)^2 + (y - 5)^2$

   (d) (3 marks) A function g defined on $\mathbb{R}^2$ such that $\nabla g(-1, 10) = 14\mathbf{i} + 3\mathbf{j}$.

   Solution: $g(x, y) = 14x + 3y$

12 marks

2. In this question, you only need to provide answers, with no justification required.

(a) (4 marks) Let $f(x,y)$ be a function defined on an open set containing $(0,0)$ which satisfies the following condition: for every $\epsilon > 0$ there is a $\delta > 0$ such that if $|x| < \delta$ and $|y| < \delta$, then $|f(x,y)| \le \epsilon$. Does it have to be true that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$?

Solution: Yes, it has to be true.

(b) (4 marks) Suppose that the function $f(x,y)$ and all its partial derivatives are continuous for all x, y, and that its second order Taylor polynomial at (a,b) is

$$p_2(x,y) = -7 + 4(x-a) - 13(x-a)^2 - 6(x-a)(y-b) + 10(y-b)^2.$$

Find the partial derivatives $f_1(a,b)$, $f_{11}(a,b)$, and $f_{12}(a,b)$.

Solution: $f_1(a,b) = 4$, $f_{11}(a,b) = -26$, and $f_{12}(a,b) = -6$.

(c) (4 marks) What is $\iint_D (xy+3)dA$, if D is the unit disc $\{(x,y) : \; x^2+y^2 \le 1\}$?

Solution: $0 + 3 \cdot \pi = 3\pi$.

12 marks 3. The surfaces $4x^2 + y^2 = z^2$ and $xy = z + 1$ intersect in a curve that contains the point $P = (2, 3, 5)$. Find the equation of the line tangent to that curve at P.

Solution:

The direction vector $\mathbf{v}$ of the line has to be tangent to both surfaces, therefore perpendicular to the normal vectors to both surfaces. The surfaces can be represented as level surfaces $f(x, y, z) = 0$ and $g(x, y, z)$, where $f(x, y, z) = 4x^2 + y^2 - z^2$ and $g(x, y, z) = xy - z$. The normal vectors at P are

$$\mathbf{n}_1 = \nabla f(2, 3, 5), \quad \mathbf{n}_2 = \nabla g(2, 3, 5).$$

We have

$$\nabla f(x, y, z) = \langle 8x, 2y, -2z\rangle, \quad \nabla g(x, y, z) = \langle y, x, -1\rangle,$$

so that

$$\mathbf{n}_1 = \langle 16, 6, -10\rangle, \quad \mathbf{n}_2 = \langle 3, 2, -1\rangle.$$

A vector perpendicular to both normals is

$$\frac{1}{2}\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 3 & -5 \\ 3 & 2 & -1 \end{vmatrix} = (-3 + 10)\mathbf{i} - (-8 + 15)\mathbf{j} + (16 - 9)\mathbf{k} = 7\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}.$$

For convenience, we will use $\mathbf{i} - \mathbf{j} + \mathbf{k}$ for the direction vector $\mathbf{v}$. The line is parallel to $\mathbf{v}$ and passes through $(2, 3, 5)$, so it has the parametric equations

$$x = 2 + t, \quad y = 3 - t, \quad z = 5 + t.$$

12 marks 4. Let $f(x,y) = x^3 + 3x^2y - y - y^3$. Find all critical points of f and classify them as local minima, maxima, or saddle points.

Solution: We have

$$f_1(x,y) = 3x^2 + 6xy, \quad f_2(x,y) = 3x^2 - 1 - 3y^2.$$

We look for points with $f_1 = f_2 = 0$. If $f_1 = 3x^2+6xy = 3x(x+2y) = 0$, then $x = 0$ or $x = -2y$. If $x = 0$, then from the condition on f_2 we get $3x^2-1-3y^2 = -1-3y^2 = 0$, so that $3y^2 = -1$, which is not possible. Therefore $x = -2y$. from the condition on f_2 we then get $3x^2 - 1 - 3y^2 = 12y^2 - 1 - 3y^2 = 0$, so that $9y^2 = 1$, $y = \pm\frac{1}{3}$. This corresponds to two critical points, $(-\frac{2}{3}, \frac{1}{3})$ and $(\frac{2}{3}, -\frac{1}{3})$.

The second order partials are $f_{11}(x,y) = 6x + 6y$, $f_{12}(x,y) = f_{21}(x,y) = 6x$, $f_{22}(x,y) = -6y$.

- At $(-\frac{2}{3}, \frac{1}{3})$,
$$\begin{vmatrix} -2 & -4 \\ -4 & -2 \end{vmatrix} = 4 - 16 < 0,$$
so by the second derivative test, f has a saddle point.
- At $(\frac{2}{3}, -\frac{1}{3})$
$$\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} = 4 - 16 < 0,$$
so by the second derivative test, f has a saddle point.

12 marks 5. Find the largest and smallest values of the function $f(x, y) = x^2 - 3x - \frac{y^2}{2}$ in the closed disc $x^2 + y^2 \leq 4$.

Solution: We first look for critical points: from $\nabla f = \langle 2x - 3, -y\rangle = 0$, we get $y = 0$ and $x = 3/2$. The corresponding value of f is

$$f(\frac{3}{2}, 0) = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}.$$

Next, we look for the largest and smallest values on the circle $x^2 + y^2 = 4$. We use Lagrange multipliers with $g(x, y) = x^2 + y^2 - 4$: $\nabla g = \langle 2x, 2y\rangle$, so that for $\nabla f = \lambda \nabla g$ we must have $2x - 3 = 2\lambda x$, $-y = 2\lambda y$. From the second equation, we have $y = 0$ or $2\lambda = -1$.

- If $y = 0$, then from the equation of the circle we get $x^2 = 4$, $x = \pm 2$, with the corresponding values of f:

$$f(2, 0) = 4 - 6 = -2, \quad f(-2, 0) = 4 + 6 = 10.$$

- If $2\lambda = -1$, then from the first equation we have $2x - 3 = -x$, $3x = 3$, $x = 1$. From the equation of the circle, we then have $y^2 = 3$, $y = \pm\sqrt{3}$. The corresponding values of f are

$$f(1, \pm\sqrt{3}) = 1 - 3 - \frac{3}{2} = -\frac{7}{2}.$$

Finally, we choose the largest and smallest of these values: the largest value is $f(-2, 0) = 4 + 6 = 10$, and the smallest value is $f(1, \pm\sqrt{3}) = 1 - 3 - \frac{3}{2} = -\frac{7}{2}$.

10 marks 6. Suppose that the functions $x(u,v)$ and $y(u,v)$ solve the equations

$$\begin{cases} u &= 3x+2e^y \\ v &= e^x - y \end{cases}$$

in some neighbourhood of the point where $(x,y)=(1,0)$. Find $\dfrac{\partial x}{\partial u}$ and $\dfrac{\partial y}{\partial u}$ at that point.

Solution: We differentiate the equations in u:

$$\begin{cases} 1 &= 3\frac{\partial x}{\partial u} + 2e^y\frac{\partial y}{\partial u} \\ \\ 0 &= e^x\frac{\partial x}{\partial u} - \frac{\partial y}{\partial u} \end{cases}$$

From the second equation, $\frac{\partial y}{\partial u} = e^x\frac{\partial x}{\partial u}$. Plugging this into the first equation, we get

$$3\frac{\partial x}{\partial u} + 2e^y e^x\frac{\partial x}{\partial u} = (3+2e^{x+y})\frac{\partial x}{\partial u} = 1,$$

$$\frac{\partial x}{\partial u} = \frac{1}{3+2e^{x+y}}, \quad \frac{\partial y}{\partial u} = \frac{e^x}{3+2e^{x+y}}.$$

Finally, plugging in $(x,y)=(1,0)$, we get that at that point,

$$\frac{\partial x}{\partial u} = \frac{1}{3+2e}, \quad \frac{\partial y}{\partial u} = \frac{e}{3+2e}.$$

8 marks 7. Evaluate the double integral $\iint_D x\,dA$, where D is the triangle with vertices $(0,0)$, $(2,1)$, $(4,0)$.

Solution:

One way to do it is to notice that $\iint_D x\,dA =$ (area of D) $\times$ (the average value of x on D). The average value of x on D is, by symmetry, 2. The area of D is $\frac{1}{2}\cdot 4\cdot 1 = 2$. Therefore $\iint_D x\,dA = 2\cdot 2 = 4$.

Alternatively, one can evaluate the iterated integral:

$$\begin{aligned}
\iint_D x\,dA &= \int_0^1 \int_{2y}^{4-2y} x dx dy = \int_0^1 \frac{x^2}{2}\Big|_{2y}^{4-2y} dy \\
&= \int_0^1 2\big((2-y)^2 - y^2\big) dy = \int_0^1 (8-8y) dy \\
&= 8y - 4y^2\Big|_0^1 = 8 - 4 = 4.
\end{aligned}$$

10 marks 8. Let D be the spatial region above the xy-plane, inside the cylinder $x^2 + y^2 = 1$, and inside the sphere $x^2 + y^2 + z^2 = 4$. Evaluate the integral $\iiint_D z dV$.

Solution: We use cylindrical coordinates:

$$\begin{aligned}
\iiint_D z dV &= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} zr\, dz\, d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} r\, \frac{z^2}{2}\Big|_0^{\sqrt{4-r^2}} d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} r\, \frac{4-r^2}{2}\, d\theta\, dr \\
&= \int_0^1 \pi(4r - r^3)\, dr \\
&= \pi\left(2r^2 - \frac{r^4}{4}\right)\Big|_0^1 = \pi\left(2 - \frac{1}{4}\right) = \frac{7\pi}{4}.
\end{aligned}$$

12 marks 9. (In this question, you have to use the definitions of partial derivatives and differentiability, but you do not have to give $\epsilon - \delta$ proofs for the limits involved.)

Let

$$f(x,y) = \begin{cases} \dfrac{x^3 - \sin^3 x}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (6 marks) Find the first order partial derivatives of f at $(0,0)$.

Solution: Using that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we get

$$\begin{aligned} f_1(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{x^3 - \sin^3 x}{x^3} \\ &= \lim_{x\to 0}\left(1 - \frac{\sin^3 x}{x^3}\right) = 1 - 1 = 0, \\ f_2(0,0) &= \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{0}{y^3} = 0. \end{aligned}$$

(b) (6 marks) Is f differentiable at $(0,0)$? Prove your answer.

Solution: Based on part (a), the linear approximation to f at $(0,0)$ is $L(x,y) = 0$. For f to be differentiable at $(0,0)$, we need

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^3-\sin^3 x}{(x^2+y^2)^{3/2}} = 0.$$

We have

$$\lim_{(x,y)\to(0,0)} \left|\frac{x^3-\sin^3 x}{(x^2+y^2)^{3/2}}\right| \leq \lim_{(x,y)\to(0,0)} \left|\frac{x^3-\sin^3 x}{x^3}\right|$$

and that limit is 0 as in part (a). Therefore the function is differentiable.