

*This final exam has **9 questions** on **11 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Last name: _____ First name: _____

Student no.: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	12	12	12	12	12	10	8	10	12	100
Score:										

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Please read the following points carefully before starting to write.

- In all questions except Q1 and 2, give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Write clearly and legibly, in complete sentences. Make sure that the logic of your argument is clear.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- If you run out of space, continue on the back of the page, on the blank page at the end, or in an additional booklet, **with clear indication on the original page of where the solution is continued.**

12 marks

1. Give examples of the following. (In this question, you do not need to justify your answers.)

(a) (3 marks) Three non-zero vectors \mathbf{u} , \mathbf{v} , \mathbf{w} such that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4$.

Solution: $\mathbf{u} = 4\mathbf{k}$, $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$ is one example. Then $\mathbf{v} \times \mathbf{w} = \mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4$.

(b) (3 marks) Three non-zero vectors \mathbf{u} , \mathbf{v} , \mathbf{w} such that $\mathbf{u} \neq \mathbf{v}$ but \mathbf{u} and \mathbf{v} have the same vector projections on \mathbf{w} .

Solution: Let $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$, $\mathbf{w} = \mathbf{k}$, then both vector projections are $\mathbf{0}$.

(c) (3 marks) A function f defined on \mathbb{R}^2 which has a global minimum at $(2, 5)$ and no other local or global maxima or minima.

Solution:

$$f(x, y) = (x - 2)^2 + (y - 5)^2$$

(d) (3 marks) A function g defined on \mathbb{R}^2 such that $\nabla g(-1, 10) = 14\mathbf{i} + 3\mathbf{j}$.

Solution: $g(x, y) = 14x + 3y$

12 marks

2. In this question, you only need to provide answers, with no justification required.

- (a) (4 marks) Let $f(x, y)$ be a function defined on an open set containing $(0, 0)$ which satisfies the following condition: for every $\epsilon > 0$ there is a $\delta > 0$ such that if $|x| < \delta$ and $|y| < \delta$, then $|f(x, y)| \leq \epsilon$. Does it have to be true that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$?

Solution: Yes, it has to be true.

- (b) (4 marks) Suppose that the function $f(x, y)$ and all its partial derivatives are continuous for all x, y , and that its second order Taylor polynomial at (a, b) is

$$p_2(x, y) = -7 + 4(x - a) - 13(x - a)^2 - 6(x - a)(y - b) + 10(y - b)^2.$$

Find the partial derivatives $f_1(a, b)$, $f_{11}(a, b)$, and $f_{12}(a, b)$.

Solution: $f_1(a, b) = 4$, $f_{11}(a, b) = -26$, and $f_{12}(a, b) = -6$.

- (c) (4 marks) What is $\iint_D (xy + 3)dA$, if D is the unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$?

Solution: $0 + 3 \cdot \pi = 3\pi$.

12 marks

3. The surfaces $4x^2 + y^2 = z^2$ and $xy = z + 1$ intersect in a curve that contains the point $P = (2, 3, 5)$. Find the equation of the line tangent to that curve at P .

Solution:

The direction vector \mathbf{v} of the line has to be tangent to both surfaces, therefore perpendicular to the normal vectors to both surfaces. The surfaces can be represented as level surfaces $f(x, y, z) = 0$ and $g(x, y, z)$, where $f(x, y, z) = 4x^2 + y^2 - z^2$ and $g(x, y, z) = xy - z$. The normal vectors at P are

$$\mathbf{n}_1 = \nabla f(2, 3, 5), \quad \mathbf{n}_2 = \nabla g(2, 3, 5).$$

We have

$$\nabla f(x, y, z) = \langle 8x, 2y, -2z \rangle, \quad \nabla g(x, y, z) = \langle y, x, -1 \rangle,$$

so that

$$\mathbf{n}_1 = \langle 16, 6, -10 \rangle, \quad \mathbf{n}_2 = \langle 3, 2, -1 \rangle.$$

A vector perpendicular to both normals is

$$\frac{1}{2}\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 3 & -5 \\ 3 & 2 & -1 \end{vmatrix} = (-3 + 10)\mathbf{i} - (-8 + 15)\mathbf{j} + (16 - 9)\mathbf{k} = 7\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}.$$

For convenience, we will use $\mathbf{i} - \mathbf{j} + \mathbf{k}$ for the direction vector \mathbf{v} . The line is parallel to \mathbf{v} and passes through $(2, 3, 5)$, so it has the parametric equations

$$x = 2 + t, \quad y = 3 - t, \quad z = 5 + t.$$

12 marks

4. Let $f(x, y) = x^3 + 3x^2y - y - y^3$. Find all critical points of f and classify them as local minima, maxima, or saddle points.

Solution: We have

$$f_1(x, y) = 3x^2 + 6xy, \quad f_2(x, y) = 3x^2 - 1 - 3y^2.$$

We look for points with $f_1 = f_2 = 0$. If $f_1 = 3x^2 + 6xy = 3x(x + 2y) = 0$, then $x = 0$ or $x = -2y$. If $x = 0$, then from the condition on f_2 we get $3x^2 - 1 - 3y^2 = -1 - 3y^2 = 0$, so that $3y^2 = -1$, which is not possible. Therefore $x = -2y$. From the condition on f_2 we then get $3x^2 - 1 - 3y^2 = 12y^2 - 1 - 3y^2 = 0$, so that $9y^2 = 1$, $y = \pm\frac{1}{3}$. This corresponds to two critical points, $(-\frac{2}{3}, \frac{1}{3})$ and $(\frac{2}{3}, -\frac{1}{3})$.

The second order partials are $f_{11}(x, y) = 6x + 6y$, $f_{12}(x, y) = f_{21}(x, y) = 6x$, $f_{22}(x, y) = -6y$.

- At $(-\frac{2}{3}, \frac{1}{3})$,

$$\begin{vmatrix} -2 & -4 \\ -4 & -2 \end{vmatrix} = 4 - 16 < 0,$$

so by the second derivative test, f has a saddle point.

- At $(\frac{2}{3}, -\frac{1}{3})$

$$\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} = 4 - 16 < 0,$$

so by the second derivative test, f has a saddle point.

12 marks

5. Find the largest and smallest values of the function $f(x, y) = x^2 - 3x - \frac{y^2}{2}$ in the closed disc $x^2 + y^2 \leq 4$.

Solution: We first look for critical points: from $\nabla f = \langle 2x - 3, -y \rangle = 0$, we get $y = 0$ and $x = 3/2$. The corresponding value of f is

$$f\left(\frac{3}{2}, 0\right) = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}.$$

Next, we look for the largest and smallest values on the circle $x^2 + y^2 = 4$. We use Lagrange multipliers with $g(x, y) = x^2 + y^2 - 4$: $\nabla g = \langle 2x, 2y \rangle$, so that for $\nabla f = \lambda \nabla g$ we must have $2x - 3 = 2\lambda x$, $-y = 2\lambda y$. From the second equation, we have $y = 0$ or $2\lambda = -1$.

- If $y = 0$, then from the equation of the circle we get $x^2 = 4$, $x = \pm 2$, with the corresponding values of f :

$$f(2, 0) = 4 - 6 = -2, \quad f(-2, 0) = 4 + 6 = 10.$$

- If $2\lambda = -1$, then from the first equation we have $2x - 3 = -x$, $3x = 3$, $x = 1$. From the equation of the circle, we then have $y^2 = 3$, $y = \pm\sqrt{3}$. The corresponding values of f are

$$f(1, \pm\sqrt{3}) = 1 - 3 - \frac{3}{2} = -\frac{7}{2}.$$

Finally, we choose the largest and smallest of these values: the largest value is $f(-2, 0) = 4 + 6 = 10$, and the smallest value is $f(1, \pm\sqrt{3}) = 1 - 3 - \frac{3}{2} = -\frac{7}{2}$.

10 marks

6. Suppose that the functions $x(u, v)$ and $y(u, v)$ solve the equations

$$\begin{cases} u &= 3x + 2e^y \\ v &= e^x - y \end{cases}$$

in some neighbourhood of the point where $(x, y) = (1, 0)$. Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ at that point.

Solution: We differentiate the equations in u :

$$\begin{cases} 1 &= 3\frac{\partial x}{\partial u} + 2e^y\frac{\partial y}{\partial u} \\ 0 &= e^x\frac{\partial x}{\partial u} - \frac{\partial y}{\partial u} \end{cases}$$

From the second equation, $\frac{\partial y}{\partial u} = e^x\frac{\partial x}{\partial u}$. Plugging this into the first equation, we get

$$3\frac{\partial x}{\partial u} + 2e^ye^x\frac{\partial x}{\partial u} = (3 + 2e^{x+y})\frac{\partial x}{\partial u} = 1,$$

$$\frac{\partial x}{\partial u} = \frac{1}{3 + 2e^{x+y}}, \quad \frac{\partial y}{\partial u} = \frac{e^x}{3 + 2e^{x+y}}.$$

Finally, plugging in $(x, y) = (1, 0)$, we get that at that point,

$$\frac{\partial x}{\partial u} = \frac{1}{3 + 2e}, \quad \frac{\partial y}{\partial u} = \frac{e}{3 + 2e}.$$

8 marks

7. Evaluate the double integral $\iint_D x \, dA$, where D is the triangle with vertices $(0, 0)$, $(2, 1)$, $(4, 0)$.

Solution:

One way to do it is to notice that $\iint_D x \, dA = (\text{area of } D) \times (\text{the average value of } x \text{ on } D)$. The average value of x on D is, by symmetry, 2. The area of D is $\frac{1}{2} \cdot 4 \cdot 1 = 2$. Therefore $\iint_D x \, dA = 2 \cdot 2 = 4$.

Alternatively, one can evaluate the iterated integral:

$$\begin{aligned} \iint_D x \, dA &= \int_0^1 \int_{2y}^{4-2y} x \, dx \, dy = \int_0^1 \left. \frac{x^2}{2} \right|_{2y}^{4-2y} dy \\ &= \int_0^1 2((2-y)^2 - y^2) \, dy = \int_0^1 (8 - 8y) \, dy \\ &= 8y - 4y^2 \Big|_0^1 = 8 - 4 = 4. \end{aligned}$$

10 marks

8. Let D be the spatial region above the xy -plane, inside the cylinder $x^2 + y^2 = 1$, and inside the sphere $x^2 + y^2 + z^2 = 4$. Evaluate the integral $\iiint_D z dV$.

Solution: We use cylindrical coordinates:

$$\begin{aligned}\iiint_D z dV &= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} zr dz d\theta dr \\ &= \int_0^1 \int_0^{2\pi} r \left. \frac{z^2}{2} \right|_0^{\sqrt{4-r^2}} d\theta dr \\ &= \int_0^1 \int_0^{2\pi} r \frac{4-r^2}{2} d\theta dr \\ &= \int_0^1 \pi(4r - r^3) dr \\ &= \pi \left(2r^2 - \frac{r^4}{4} \right) \Big|_0^1 = \pi \left(2 - \frac{1}{4} \right) = \frac{7\pi}{4}.\end{aligned}$$

12 marks

9. (In this question, you have to use the definitions of partial derivatives and differentiability, but you do not have to give $\epsilon - \delta$ proofs for the limits involved.)

Let

$$f(x, y) = \begin{cases} \frac{x^3 - \sin^3 x}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) (6 marks) Find the first order partial derivatives of f at $(0, 0)$.

Solution: Using that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, we get

$$\begin{aligned} f_1(0, 0) &= \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x^3 - \sin^3 x}{x^3} \\ &= \lim_{x \rightarrow 0} \left(1 - \frac{\sin^3 x}{x^3} \right) = 1 - 1 = 0, \\ f_2(0, 0) &= \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0}{y^3} = 0. \end{aligned}$$

- (b) (6 marks) Is f differentiable at $(0, 0)$? Prove your answer.

Solution: Based on part (a), the linear approximation to f at $(0, 0)$ is $L(x, y) = 0$. For f to be differentiable at $(0, 0)$, we need

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - L(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - \sin^3 x}{(x^2 + y^2)^{3/2}} = 0.$$

We have

$$\lim_{(x,y) \rightarrow (0,0)} \left| \frac{x^3 - \sin^3 x}{(x^2 + y^2)^{3/2}} \right| \leq \lim_{(x,y) \rightarrow (0,0)} \left| \frac{x^3 - \sin^3 x}{x^3} \right|$$

and that limit is 0 as in part (a). Therefore the function is differentiable.