MATH 317 MIDTERM 1, OCTOBER 2011: SOLUTIONS

1. The plane $z = x+2$ intersects the cone $z^2 = x^2+y^2$ in a curve. Find the equation of the curve. (Either vector or parametric equations are fine.)

Along the intersection curve we have $(x+2)^2 = x^2+y^2$, $x^2+4x+4 = x^2+y^2$, $4x = y^2-4$, $x = (y^2/4)-1$. Let $y = 2t$, then $x = t^2-1$, $z = x+2 = t^2+1$.

(The vector equation is $\mathbf{r}(t) = \langle t^2-1, 2t, t^2+1\rangle$. If we let $y = t$ instead, we get $x = (t^2/4)-1$, $z = (t^2/4)+4$. Substitutions $x = t$ and $z = t$ can also work, but then two different formulas are required for two parts of the curve.)

2. A curve in $\mathbb{R}^3$ is given by the vector equation $\mathbf{r}(t) = \langle t^3, e^{t-1}, t+1\rangle$. Find the parametric equations of the tangent line to the curve at the point $(1,1,2)$.

The point $(1,1,2)$ corresponds to $t = 1$. We have $\mathbf{r}'(t) = \langle 3t^2, e^{t-1}, 1\rangle$, $\mathbf{r}'(1) = \langle 3,1,1\rangle$. The line is parallel to this vector and passes through $(1,1,2)$, so the parametric equations are $x = 1+3t$, $y = 1+t$, $z = 2+t$.

3. A curve in $\mathbb{R}^3$ is given by the parametric equations

$$x = \frac{4}{3}t^3 + t,\ \ y = t^3, z = \frac{t^2}{\sqrt{2}}.$$

Find the length of the curve from $t = 0$ to $t = 1$. (Evaluate the integral. The answer should be a number.)

We have

$$\mathbf{r}(t) = \langle \frac{4}{3}t^3 + t, t^3, \frac{t^2}{\sqrt{2}}\rangle,\ \ \mathbf{r}'(t) = \langle 4t^2+1, 3t^2, \sqrt{2}t\rangle,$$

$$\begin{aligned}|\mathbf{r}'(t)| &= \big((4t^2+1)^2 + (3t^2)^2 + (\sqrt{2}t)^2\big)^{1/2} \\ &= \big(16t^4 + 8t^2 + 1 + 9t^4 + 2t^2\big)^{1/2} \\ &= \big(25t^4 + 10t^2 + 1\big)^{1/2} = 5t^2 + 1,\end{aligned}$$

$$\text{length} = \int_0^1 (5t^2+1)dt = \frac{5}{3}t^3 + t\Big|_0^1 = \frac{5}{3} + 1 = \frac{8}{3}$$

4. A bug is flying in $\mathbb{R}^3$ along a curve $\mathbf{r}(t)$. Its position at time $t = 0$ is $(1,2,-1)$, and its velocity at time t is $\mathbf{v}(t) = 2t\mathbf{i} + (t-1)\mathbf{j}$.

(a) Find $\mathbf{r}(1)$ (the bug's position at time $t = 1$).

$$\begin{aligned}\mathbf{r}(1) &= \int_0^1 \Big(2t\mathbf{i} + (t-1)\mathbf{j}\Big)dt + (1,2,-1) \\ &= t^2\mathbf{i} + (\frac{t^2}{2} - t)\mathbf{j}\Big|_0^1 + (1,2,-1) \\ &= (1, \frac{1}{2} - 1, 0) + (1,2,-1) = (2, \frac{3}{2}, -1)\end{aligned}$$

(b) Find the bug's speed and acceleration at time t.

$$\text{Speed}: \ v(t) = |\mathbf{v}(t)| = \sqrt{4t^2 + (t-1)^2} = \sqrt{4t^2 + t^2 - 2t + 1} = \sqrt{5t^2 - 2t + 1}$$
$$\text{Acceleration}: \ \mathbf{a}(t) = \mathbf{v}'(t) = 2\mathbf{i} + \mathbf{j}$$

(c) For what t is the tangential component of the acceleration equal to 0?

The tangential component is

$$a_T = v'(t) = \frac{10t - 2}{2\sqrt{5t^2 - 2t + 1}} = \frac{5t - 1}{\sqrt{5t^2 - 2t + 1}}.$$

This is 0 when $t = 1/5$.

5. A curve in $\mathbb{R}^3$ has the vector equation $\mathbf{r}(t) = \langle 2\sin t, t, -2\cos t\rangle$. Find the unit tangent, principal normal and binormal vectors, and the curvature at a general point on the curve.

$$\mathbf{r}'(t) = \langle 2\cos t, 1, 2\sin t\rangle, \ |\mathbf{r}'(t)| = \sqrt{4\cos^2 t + 1 + 4\sin^2 t} = \sqrt{5}$$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{5}}\langle 2\cos t, 1, 2\sin t\rangle$$
$$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -2\sin t, 0, 2\cos t\rangle, \ |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{4\sin^2 t + 4\cos^2 t} = \frac{2}{\sqrt{5}}$$
$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{2/\sqrt{5}}{\sqrt{5}} = \frac{2}{5}$$
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1/\sqrt{5}}{2/\sqrt{5}}\langle -2\sin t, 0, 2\cos t\rangle = \langle -\sin t, 0, \cos t\rangle$$
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{5}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos t & 1 & 2\sin t \\ -\sin t & 0 & \cos t \end{vmatrix} = \frac{1}{\sqrt{5}}\langle \cos t, -2, \sin t\rangle$$