Final Examination — December 19, 2015 Duration: 2.5 hours

*This test has **8 questions** on **9 pages**, for a total of 80 points.*

Dr. G. Slade

- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space, *with clear indication on the original page* that your solution is continued elsewhere.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: __________ Last Name: __________

Student-No: __________ Section: __________

Signature: __________

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	10	10	10	10	10	10	80
Score:									

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)?(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

10 marks 1. Define the following.

(a) X is a *metric space* with *metric* d.

Solution: X is a set and $d : X \times X \to \mathbb{R}$ is a function such that, for all $x, y, z \in X$,

- $d(x, y) \geq 0$ with equality if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$.

(b) The subset E of a metric space X is *dense*.

Solution: Every $x \in X$ is either a point in E or a limit point of E.
Or: every open subset of X contains a point in E.

(c) An *open cover* of a subset E of a metric space X.

Solution: An open cover is a collection of open sets $\{O_\alpha\}$ such that $E \subset \cup_\alpha O_\alpha$.

(d) A *connected* subset E of a metric space X.

Solution: A set E which cannot be written as a union $E = A \cup B$ with $A \cap \overline{B} = \overline{A} \cap B = \varnothing$, with A, B nonempty.

(e) The series $\sum_{n=1}^{\infty} a_n$ *converges and equals* A (here each a_n is a complex number).

Solution: Let $s_n = \sum_{n=0}^{n} a_k$. It means that the sequence $\{s_n\}$ converges to A.

2. Let $A_1, A_2, \ldots$ be subsets of a metric space X.

6 marks (a) For $n = 1, 2, \ldots$, let $B_n = \cup_{i=1}^n A_i$. Prove that $\overline{B_n} = \cup_{i=1}^n \overline{A_i}$.

Solution: Since $\cup_{i=1}^n \overline{A_i}$ is a finite union of closed sets, it is closed. Since $\overline{A_i} \supset A_i$, we have $\cup_{i=1}^n \overline{A_i} \supset B_n$. The closure of a set is contained in every closed set that contains it, so $\cup_{i=1}^n \overline{A_i} \supset \overline{B_n}$.
On the other hand, suppose that $x \in \cup_{i=1}^n \overline{A_i}$. If $x \in \cup_{i=1}^n A_i = B_n$ then $x \in \overline{B_n}$. So it suffices to consider the case where x is a limit point of one of the sets A_i, say A_k. Then every neighbourhood of x contains a point in $A_k \subset B_n$, and hence is a limit point of B_n, hence $x \in \overline{B_n}$. This shows that $\cup_{i=1}^n \overline{A_i} \subset \overline{B_n}$.

2 marks (b) Let $B = \cup_{i=1}^\infty A_i$. Prove that $\overline{B} \supset \cup_{i=1}^\infty \overline{A_i}$.

Solution: The proof from part (a) applies directly, as follows.
Suppose that $x \in \cup_{i=1}^\infty \overline{A_i}$. If $x \in \cup_{i=1}^\infty A_i = B$ then $x \in \overline{B}$. So it suffices to consider the case where x is a limit point of one of the sets A_i, say A_k. Then every neighbourhood of x contains a point in $A_k \subset B$, and hence is a limit point of B, hence $x \in \overline{B}$. This shows that $\cup_{i=1}^\infty \overline{A_i} \subset \overline{B}$.

2 marks (c) Give an example to show that the inclusion in part (b) can be proper.

Solution: For $X = \mathbb{R}$, let $A_i = (\frac{1}{i}, \infty)$. Then $B = (0, \infty)$, $\overline{B} = [0, \infty)$, $\overline{A_i} = [\frac{1}{i}, \infty)$, and $\cup_{i=1}^\infty \overline{A_i} = (0, \infty)$.

3. Let $B \subset \mathbb{R}^k$ be nonempty and compact, and $a \in B^c$. The metric on $\mathbb{R}^k$ is the usual metric $d(x, y) = |x - y|$. Recall that the distance $d(a, B)$ from a to B is defined by $d(a, B) = \inf\{d(a, b) : b \in B\}$.

8 marks

(a) Prove that there exists $b \in B$ such that $d(a, b) = d(a, B)$.

Solution: Since $d(a, B) = \inf\{d(a, b) : b \in B\}$, there exists $b_n \in B$ such that $d(a, b_n) < d(a, B) + \frac{1}{n}$, for all $n \in \mathbb{N}$. Let $E = \{b_n : n \in \mathbb{N}\}$. By Theorem 2.41, E has a limit point in E, call it $b' \in E$. Let $m \in \mathbb{N}$ be given. There exists $M \geq m$ such that $d(b', b_M) < \frac{1}{m}$ (we can insist on $M \geq m$ because every neighbourhood of b' contains infinitely many points of E by Theorem 2.20). By the triangle inequality,

$$d(a, b') \leq d(a, b_M) + d(b_M, b') < d(a, B) + \frac{1}{M} + \frac{1}{m} \leq d(a, B) + \frac{2}{m}.$$

Since m is arbitrary, this means $d(a, b') \leq d(a, B)$. However, by definition of $d(a, B)$ and the fact that $b' \in B$, $d(a, B) \leq d(a, b')$. Therefore, $d(a, B) = d(a, b')$.

2 marks

(b) Is the b in part (a) unique? Give a short proof if so, or a counterexample if not.

Solution: It is not unique. Let $\vec{e}$ be one of the standard unit vectors in $\mathbb{R}^k$, let $B = \{\vec{e}, -\vec{e}\}$, and let $a = \vec{0}$. Then

$$d(\vec{0}, B) = 1 = d(\vec{0}, \vec{e}) = d(\vec{0}, -\vec{e}).$$

3 marks 4. (a) Let $\{s_n\}$ be a sequence in $\mathbb{R}$. Recall the definitions $\liminf_{n\to\infty} s_n = \sup_{n\geq 1} \inf_{m\geq n} s_m$ and $\limsup_{n\to\infty} s_n = \inf_{n\geq 1} \sup_{m\geq n} s_m$. Consider the sequence $a_1, a_2, a_3, \ldots$ in $\mathbb{Z}$ which starts with $1, 6, 2, 12, 4, 24, 8, 48, 16, 96, \ldots$ and continues this pattern. Determine each of the following six values, or state that they do not exist:
$L_1 = \liminf_{n\to\infty} a_{n+1}/a_n$, $L_2 = \limsup_{n\to\infty} a_{n+1}/a_n$, $L_3 = \lim_{n\to\infty} a_{n+1}/a_n$,
$M_1 = \liminf_{n\to\infty} a_{n+2}/a_n$, $M_2 = \limsup_{n\to\infty} a_{n+2}/a_n$, $M_3 = \lim_{n\to\infty} a_{n+2}/a_n$.

Solution: Since a_{n+1}/a_n alternates between 6 and $\frac{1}{3}$, $L_1 = \frac{1}{3}$, $L_2 = 6$, and L_3 does not exist.
Since a_{n+2}/a_n is constantly equal to 2, $M_1 = M_2 = M_3 = 2$.

3 marks (b) Let X be a metric space and suppose that the sequence $\{s_n\}$ in X converges. Prove that $\{s_n\}$ is a Cauchy sequence.

Solution: Let s be the limit, and let $\epsilon > 0$. Choose N such that $n \geq N$ implies that $d(s, s_n) < \epsilon/2$. Then, if $m, n \geq N$, it follows from the triangle inequality that

$$d(s_m, s_n) \leq d(s, s_m) + d(s, s_n) < \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves that the sequence is a Cauchy sequence.

4 marks (c) Let X be the set of all sequences of real numbers for which only finitely many members of the sequence are nonzero; thus an element $\vec{x} \in X$ is a sequence $\vec{x} = \{x_i\}$ with each $x_i \in \mathbb{R}$ and with only finitely many $x_i \neq 0$. A metric is defined on X by $d(\vec{x}, \vec{y}) = [\sum_{i=1}^{\infty} |x_i - y_i|^2]^{1/2}$ (you need not prove that this is a metric). Consider the sequence $\{\vec{x}_n\}$ in X defined by $\vec{x}_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots)$. Prove that $\{\vec{x}_n\}$ is a Cauchy sequence in X.

Solution: By definition, for $n > m$,

$$d(\vec{x}_n, \vec{x}_m) = \left[\sum_{i=m+1}^{n} \frac{1}{i^2}\right]^{1/2}.$$

Let $\epsilon > 0$. Since $\sum_i \frac{1}{i^2}$ is a convergent series, its sequence of partial sums is a Cauchy sequence, and hence the right-hand side above can be made less than ϵ by taking $n > m \geq N$. This proves that $\{\vec{x}_n\}$ is a Cauchy sequence.

10 marks

5. Let $E \subset \mathbb{R}$ be a bounded set, and suppose that $f : E \to \mathbb{R}$ is uniformly continuous. Prove that $f(E)$ is bounded.

Solution: Choose $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < 1$. Fix a bounded interval $[a, b]$ containing E. Choose N such that $\frac{b-a}{N} < \delta$, and let $I_1, \ldots, I_N$ be the N consecutive closed intervals of length δ whose union is $[a, b]$. For each k such that $I_k \cap E \neq \varnothing$, choose $x_k \in I_k \cap E$. Then $|f(x) - f(x_k)| < 1$ for all $x \in I_k$. Set $M = \max_{k=1,\ldots,N}(1 + |f(x_k)|)$. Given $x \in E$, let k be such that $x \in I_k$. Then

$$|f(x)| \leq |f(x_k)| + |f(x) - f(x_k)| \leq |f(x_k)| + 1 \leq M,$$

so $f(E)$ is bounded.

5 marks 6. (a) Does the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converge or diverge?

Solution: We apply the root test:

$$\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^{n} = \frac{1}{(1+1/n)^n} \to e^{-1} < 1.$$

This implies convergence.

5 marks (b) For which values of $z \in \mathbb{C}$ does the series $\sum_{n=0}^{\infty} \frac{1}{1+2^{n^2}} z^n$ converge and for which values of z does it diverge?

Solution: We apply the ratio test. For all $z \in \mathbb{C}$,

$$\frac{|z^{n+1}|}{1+2^{(n+1)^2}} \frac{1+2^{n^2}}{|z^n|} = |z| \frac{1+2^{n^2}}{1+2^{(n+1)^2}} = |z| \frac{2^{-n^2}+1}{2^{-n^2}+2^{(n+1)^2-n^2}} = |z| \frac{2^{-n^2}+1}{2^{-n^2}+2^{2n+1}} \to 0.$$

This implies convergence for all $z \in \mathbb{C}$.

10 marks

7. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous, that $f(0) = 0$, and that f is differentiable on $(0, \infty)$ with f' monotonically increasing. Let $h(x) = x^{-1} f(x)$. Prove that h is monotonically increasing.

Solution: It suffices to show that $h'(x) \geq 0$ for all $x \in (0, \infty)$. By the quotient rule,

$$h'(x) = \frac{xf'(x) - f(x)}{x^2},$$

so it suffices to prove that $xf'(x) \geq f(x)$, i.e., that $f'(x) \geq x^{-1} f(x)$. By the mean value theorem, there exists $c \in (0, x)$ such that

$$\frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c).$$

It therefore suffices if $f'(x) \geq f'(c)$, but this follows from the monotonicity of f' together with $0 < c < x$.

10 marks 8. Use Taylor's Theorem to approximate $\sqrt{e}$ to within an error of magnitude at most 10^{-3}. (It is sufficient to express your approximation as a sum of rational numbers, it is not necessary to do the arithmetic to simplify fully.)

Solution: By Taylor's Theorem about zero, for $n \geq 1$ there is a $c \in (0, 1/2)$ such that

$$e^{1/2} = \sum_{k=0}^{n-1} \frac{1}{k!} \frac{1}{2^k} + \frac{e^c}{n!} \frac{1}{2^n}.$$

We want the remainder term to be at most 10^{-3}. Since $e < 3$ and hence $e^c < e^{1/2} < 2$, it is sufficient if

$$\frac{2}{n!} \frac{1}{2^{n-1}} \leq 10^{-3}, \quad \text{or} \quad n!2^n \geq 2000.$$

For $n = 1, 2, 3, 4, 5$, the values of $n!2^{n-1}$ are equal to $1, 4, 24, 192, 1920$, so $n = 5$ is enough, and our approximation is

$$e^{1/2} \approx \sum_{k=0}^{4} \frac{1}{k!} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{2!} \frac{1}{2^2} + \frac{1}{3!} \frac{1}{2^3} + \frac{1}{4!} \frac{1}{2^4}.$$