

**Midterm      October 21, 2016      Duration: 50 minutes***This test has 5 questions on 6 pages, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)
- Use the back of the previous page or the blank pages at the end if you need more space for work.

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student No.: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	5	Total
Points:	9	9	5	6	11	40
Score:						

**Student Conduct during Examinations**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)?(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

**9 marks**

1. Give examples of the following. You need not provide justification.

(a) An ordered field which does not have the least upper bound property.

**Solution:** The rational numbers  $\mathbb{Q}$ .

(b) A sequence that has infinitely many subsequential limits.

**Solution:**

Possible examples are: the sequence of all rational numbers; or the sequence  $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots\}$ ; or many others.

(c) A field  $F$  so that no order can be defined on  $F$  which would make it an ordered field.

**Solution:**  $F = \mathbb{C}$ ; or  $F = \{0, 1, \dots, p-1\}$ , where  $p$  is prime and arithmetic is mod  $p$ .

**9 marks**

2. Prove directly from the definition that  $\lim_{n \rightarrow \infty} \frac{n^2 - 1}{2n^2 + n} = \frac{1}{2}$ .

**Solution:** Note first that

$$\begin{aligned} \left| \frac{n^2 - 1}{2n^2 + n} - \frac{1}{2} \right| &= \left| \frac{n^2 - 1 - n^2 - (n/2)}{2n^2 + n} \right| \\ &= \frac{(n/2) + 1}{2n^2 + n} < \frac{2n}{2n^2} = n^{-1}. \end{aligned}$$

Let  $\varepsilon > 0$ . Assume  $n > N \equiv \varepsilon^{-1}$ . Then by the above we have

$$\left| \frac{n^2 - 1}{2n^2 + n} - \frac{1}{2} \right| < n^{-1} < \varepsilon.$$

This proves that that  $\lim_{n \rightarrow \infty} \frac{n^2 - 1}{2n^2 + n} = \frac{1}{2}$ .

5 marks

3. Let  $A$  and  $B$  be non-empty sets of real numbers such that for any  $a \in A$  and  $b \in B$ ,  $a \leq b$ . Prove that  $\sup A \leq \inf B$  (and that both of these quantities exist).

**Solution:** Let  $b \in B$ . Then  $b$  is an upper bound for  $A$ . Therefore  $\sup A$  exists and  $\sup A \leq b$ . Hence  $\sup A$  is a lower bound for  $B$ . Therefore  $\inf B$  exists and  $\inf B \geq \sup A$ .

6 marks

4. Let  $S$  be the set of integer-valued sequences converging to 0. Prove that  $S$  is countable.

**Hint:** What can you say about any sequence  $\{x_n\} \in S$ ?

**Solution:** Let  $\{x_n\} \in S$ . Then  $x_n \rightarrow 0$  and so, taking  $\varepsilon = 1/2$ , we see there is a natural number  $N$  s.t.  $n > N$  implies  $|x_n| < 1/2$  which means  $x_n = 0$  for all  $n > N$ . Therefore if  $S_N = \{\{x_n\} \in S : x_n = 0 \text{ for all } n > N\}$ , then  $S = \cup_{N=1}^{\infty} S_N$ . Hence it suffices to show that for a fixed  $N \in \mathbb{N}$ ,  $S_N$  is countable. Noting that  $\mathbb{Z}^N$  is countable (it is a finite Cartesian product of countable sets), we see that it suffices to show  $\mathbb{Z}^N \sim S_N$ . To see this, define  $\phi : \mathbb{Z}^N \rightarrow S_N$  by  $\phi(x_1, \dots, x_N) = \{x_n\}$ , where  $x_n$  is defined to be 0 for  $n > N$ . If  $\{x_n\} \in S_N$ , then  $\phi(x_1, \dots, x_N) = \{x_n\}$  (because the original sequence is 0 for  $n > N$ ) and so  $\phi$  is onto.  $\phi$  is trivially one-to-one. Hence we conclude  $\mathbb{Z}^N \sim S_N$ , as required.

5. Let  $X = (0, 1)$ . Define a function  $d : X \times X \rightarrow [0, \infty)$  by

$$d(x, y) = \begin{cases} |x - y| & \text{if } x - y \in \mathbb{Q} \\ 1 & \text{if } x - y \notin \mathbb{Q} \end{cases}$$

5 marks

(a) Prove that  $d$  is a metric on  $X$ .

**Solution:**  $d(x, y) = d(y, x)$  (because  $x - y \in \mathbb{Q}$  iff  $y - x \in \mathbb{Q}$ ) and  $d(x, x) = 0$  is obvious. Assume  $d(x, y) = 0$ . Then we must have  $x - y \in \mathbb{Q}$  and so  $d(x, y) = |x - y| = 0$ , implying that  $x = y$ .

It remains to prove the triangle inequality, so let  $x, y, z \in X$ . Note that  $d(x, y) \leq 1$  for all  $x, y \in X$ . If either  $x - y$  or  $y - z$  is irrational then  $d(x, y) + d(y, z) \geq 1 \geq d(x, z)$ . So assume that both  $x - y$  and  $y - z$  are rational. Then the ordinary triangle inequality implies

$$d(x, y) + d(y, z) = |x - y| + |y - z| \geq |x - z| = d(x, z),$$

where the last equality holds because  $(x - z) = (x - y) + (y - z)$  is the sum of two rational numbers and hence is rational.

6 marks

(b) Let  $E = (0, 1/2)$ . Is  $E$  an open set in this metric? Is it closed? Prove your answers.

**Solution:**  $E$  is open.

Proof. Let  $x \in E$  and set  $r = \min(x, \frac{1}{2} - x) \in (0, \frac{1}{2})$ . Then

$$N_r(x) = \{y \in X : d(x, y) < r\} = \{x \in X : y - x \in \mathbb{Q}, |x - y| < r\} \subset (0, 1/2),$$

where the last inclusion follows from elementary arithmetic and the choice of  $r$ . This implies  $E$  is open.

$E$  is not closed.

It suffices to show that  $1/2 \in \bar{E}$  since this would show that  $E \neq \bar{E}$ . Let  $r \in (0, 1/2)$  and choose a rational number  $q$  in  $(1/2 - r, 1/2) \subset E$ . Then  $\frac{1}{2} - q \in \mathbb{Q}$  and so  $d(\frac{1}{2}, q) = |\frac{1}{2} - q| < r$ . This shows  $q \in E \cap N_r(1/2)$  and so  $E \cap N_r(1/2)$  is non-empty for any  $r > 0$  (taking  $r \geq 1/2$  will only make this set larger), and hence shows  $1/2 \in \bar{E}$ , as required.