**Midterm October 21, 2016 Duration: 50 minutes**

*This test has **5 questions** on **6 pages**, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)
- Use the back of the previous page or the blank pages at the end if you need more space for work.

First Name: ______________________ Last Name: ______________________

Student No.: ______________________ Section: ______________________

Signature: ____________________________________________

| Question: | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Points: | 9 | 9 | 5 | 6 | 11 | 40 |
| Score: | | | | | | |

$\boxed{\text{9 marks}}$ 1. Give examples of the following. You need not provide justification.

(a) An ordered field which does not have the least upper bound property.

> **Solution:** The rational numbers $\mathbb{Q}$.

(b) A sequence that has infinitely many subsequential limits.

> **Solution:**
> Possible examples are: the sequence of all rational numbers; or the sequence $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots\}$; or many others.

(c) A field $F$ so that no order can be defined on $F$ which would make it an ordered field.

> **Solution:** $F = \mathbb{C}$; or $F = \{0, 1, \ldots, p - 1\}$, where $p$ is prime and arithmetic is mod $p$.

$\boxed{\text{9 marks}}$ 2. Prove directly from the definition that $\lim\limits_{n\to\infty} \dfrac{n^2-1}{2n^2+n} = \dfrac{1}{2}$.

**Solution:** Note first that

$$\left|\frac{n^2-1}{2n^2+n} - \frac{1}{2}\right| = \left|\frac{n^2-1-n^2-(n/2)}{2n^2+n}\right|$$
$$= \frac{(n/2)+1}{2n^2+n} < \frac{2n}{2n^2} = n^{-1}.$$

Let $\varepsilon > 0$. Assume $n > N \equiv \varepsilon^{-1}$. Then by the above we have

$$\left|\frac{n^2-1}{2n^2+n} - \frac{1}{2}\right| < n^{-1} < \varepsilon.$$

This proves that that $\lim_{n\to\infty} \frac{n^2-1}{2n^2+n} = \frac{1}{2}$.

$\boxed{\text{5 marks}}$ 3. Let $A$ and $B$ be non-empty sets of real numbers such that for any $a \in A$ and $b \in B$, $a \leq b$. Prove that $\sup A \leq \inf B$ (and that both of these quantities exist).

**Solution:** Let $b \in B$. Then $b$ is an upper bound for $A$. Therefore $\sup A$ exists and $\sup A \leq b$. Hence $\sup A$ is a lower bound for $B$. Therefore $\inf B$ exists and $\inf B \geq \sup A$.

6 marks 4. Let $S$ be the set of integer-valued sequences converging to 0. Prove that $S$ is countable.

**Hint:** What can you say about any sequence $\{x_n\} \in S$?

**Solution:** Let $\{x_n\} \in S$. Then $x_n \to 0$ and so, taking $\varepsilon = 1/2$, we see there is a natural number $N$ s.t. $n > N$ implies $|x_n| < 1/2$ which means $x_n = 0$ for all $n > N$. Therefore if $S_N = \{\{x_n\} \in S : x_n = 0 \text{ for all } n > N\}$, then $S = \cup_{N=1}^{\infty} S_N$. Hence it suffices to show that for a fixed $N \in \mathbb{N}$, $S_N$ is countable. Noting that $\mathbb{Z}^N$ is countable (it is a finite Cartesian product of countable sets), we see that it suffices to show $\mathbb{Z}^N \sim S_N$. To see this, define $\phi : \mathbb{Z}^N \to S_N$ by $\phi(x_1, \ldots, x_N) = \{x_n\}$, where $x_n$ is defined to be 0 for $n > N$. If $\{x_n\} \in S_N$, then $\phi(x_1, \ldots, x_N) = \{x_n\}$ (because the orginal sequence is 0 for $n > N$) and so $\phi$ is onto. $\phi$ is trivially one-to-one. Hence we conclude $\mathbb{Z}^N \sim S_N$, as required.

5. Let $X = (0, 1)$. Define a function $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} |x - y| & \text{if } x - y \in \mathbb{Q} \\ 1 & \text{if } x - y \notin \mathbb{Q} \end{cases}$$

5 marks (a) Prove that $d$ is a metric on $X$.

**Solution:** $d(x, y) = d(y, x)$ (because $x - y \in \mathbb{Q}$ iff $y - x \in \mathbb{Q}$) and $d(x, x) = 0$ is obvious. Assume $d(x, y) = 0$. Then we must have $x - y \in \mathbb{Q}$ and so $d(x, y) = |x - y| = 0$, implying that $x = y$.

It remains to prove the triangle inequality, so let $x, y, z \in X$. Note that $d(x, y) \leq 1$ for all $x, y \in X$. If either $x - y$ or $y - z$ is irrational then $d(x, y) + d(y, z) \geq 1 \geq d(x, z)$. So assume that both $x - y$ and $y - z$ are rational. Then the ordinary triangle inequality implies

$$d(x, y) + d(y, z) = |x - y| + |y - z| \geq |x - z| = d(x, z),$$

where the last equality holds because $(x - z) = (x - y) + (y - z)$ is the sum of two rational numbers and hence is rational.

6 marks (b) Let $E = (0, 1/2)$. Is $E$ an open set in this metric? Is it closed? Prove your answers.

**Solution:** $E$ is open.

Proof. Let $x \in E$ and set $r = \min(x, \frac{1}{2} - x) \in (0, \frac{1}{2})$. Then

$$N_r(x) = \{y \in X : d(x, y) < r\} = \{x \in X : y - x \in \mathbb{Q}, |x - y| < r\} \subset (0, 1/2),$$

where the last inclusion follows from elementary arithmetic and the choice of $r$. This implies $E$ is open.

$E$ is not closed.

It suffices to show that $1/2 \in \bar{E}$ since this would show that $E \neq \bar{E}$. Let $r \in (0, 1/2)$ and choose a rational number $q$ in $(1/2 - r, 1/2) \subset E$. Then $\frac{1}{2} - q \in \mathbb{Q}$ and so $d(\frac{1}{2}, q) = |\frac{1}{2} - q| < r$. This shows $q \in E \cap N_r(1/2)$ and so $E \cap N_r(1/2)$ is non-empty for any $r > 0$ (taking $r \geq 1/2$ will only make this set larger), and hence shows $1/2 \in \bar{E}$, as required.