Practice Midterm **October 16, 2016** **Duration: 50 minutes**

*This test has **5 questions** on **8 pages**, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)

First Name: ______________________ Last Name: ______________________

Student No.: ______________________ Section: ______________________

Signature: ____

Question:	1	2	3	4	5	Total
Points:	10	10	4	8	8	40
Score:						

1. Let $\{s_n\}$ be a sequence in $\mathbb{C}$.

2 marks (a) Define what it means to say that $\lim_{n\to\infty} s_n = s$.

Solution: For every $\epsilon > 0$ there is an N such that if $n \geq N$ then $|s_n - s| < \epsilon$.

3 marks (b) Suppose $\lim_{n\to\infty} s_n = s$. Prove that the sequence $\{s_n\}$ is bounded.

Solution: Take $\epsilon = 1$ in part (a). Then there is an N_1 such that

$$|s_n| \leq |s| + |s_n - s| \leq |s| + 1 \quad \text{for } n \geq N_1.$$

Let $M = \max\{|s_1|, \ldots, |s_{N_1}|, |s| + 1\}$. Then $|s_n| \leq M$ for all $n \geq 1$, so the sequence is bounded.

5 marks (c) Let $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$. Suppose $\lim_{n\to\infty} s_n = s$. Prove that $\lim_{n\to\infty} \sigma_n = s$.

Solution: By the triangle inequality,

$$|\sigma_n - s| = \frac{1}{n}|s_1 + \cdots + s_n - ns| = \frac{1}{n}|(s_1 - s) + \cdots + (s_n - s)| \le \frac{1}{n}\Big(|s_1 - s| + \cdots + |s_n - s|\Big).$$

Fix $\epsilon > 0$, and choose N as in part (a). Then $|s_m - s| < \epsilon$ for $m \ge N$, and hence

$$|\sigma_n - s| \le \frac{1}{n}\Big(|s_1 - s| + \cdots + |s_N - s|\Big) + \frac{n-N}{n}\epsilon$$

for $n \ge N$. By part (b), there is an M such that $|s_k - s| \le |s_k| + |s| \le 2M$ for all k. Therefore,

$$|\sigma_n - s| \le \frac{1}{n}2MN + \epsilon \le \epsilon + \epsilon = 2\epsilon,$$

provided we choose $n \ge \max(N, MN\epsilon^{-1})$. This proves that $\sigma_n \to s$.

2 marks 2. (a) Suppose $E \subset \mathbb{R}$ is nonempty and bounded above. Define precisely what it means to say that $x = \sup E$.

Solution: It means that (i) x is an upper bound for E, i.e., that $p \le x$ for all $p \in E$, and (ii) if y is an upper bound for E then $x \le y$.

8 marks

(b) Let A and B be non-empty subsets of $[0, \infty)$ which are bounded above, i.e., bounded subsets of non-negative real numbers. Let $AB = \{xy : x \in A,\, y \in B\}$. Prove that $\sup(AB) = (\sup A)(\sup B)$.

Solution: Let $a = \sup A$, $b = \sup B$, $c = \sup(AB)$.
Let $z \in AB$. Then $z = xy$ for some $x \in A$ and $y \in B$. Since $x \leq a$ and $y \leq b$, $z \leq ab$, so ab is an upper bound for AB and hence $c \leq ab$.
For the reverse inequality note first that if $a = 0$ or $b = 0$, then $c \geq ab$ is immediate. So assume wlog that $a, b > 0$. Let $0 < \varepsilon < \min(a, b)$, there exist $x \in A$ and $y \in B$ such that $x > a - \varepsilon$ and $y > b - \varepsilon$ (since otherwise $a - \varepsilon$ would be an upper bound for A, contradicting $a = \sup A$, and similarly for B). Then we find that $xy > (a - \varepsilon)(b - \varepsilon) > ab - \varepsilon(a + b)$, and hence $c > ab - \varepsilon(a + b)$. This holds for all $\varepsilon > 0$, so we must have $c \geq ab$.

4 marks

3. Suppose that A, B are subsets of a metric space X. We say that A is dense in B iff B is contained in the closure of A. Let $C \subset X$, and suppose that A is dense in B, and that B is dense in C. Prove that A is dense in C.

Solution: We have $B \subset \bar{A}$ and $C \subset \bar{B}$.

Therefore $C \subset \bar{B} \subset \bar{\bar{A}} = \bar{A}$, where the last equality holds since $\bar{A}$ is closed. This proves $C \subset \bar{A}$, that is, A is dense in C.

8 marks 4. True or False. If True, give a proof; if False, provide a counter-example.

(a) The set of infinite subsets of a countable set is uncountable.

Solution: TRUE.
Let C be a countable set and for each natural number N, let F_N be the set of subsets of C with cardinality at most N. Define $\phi : C^N \to F_N$ by $\phi(x_1, \ldots, x_N) = \{x_i : i \leq N\}$. Note that this set has cardinality at most N and so is in F_N. If $A \in F_N$, then for some $k \leq N$ and distinct $a_1, \ldots, a_k$ in C, $F_N = \{a_1, \ldots, a_k\} = \phi(a_1, \ldots, a_k, a_k, \ldots, a_k) \in Range(\phi)$. Hence ϕ is onto. C^N is countable because C is, and therefore there is a bijection f from $\mathbb{N}$ onto C^N. Therefore $\phi \circ f : \mathbb{N} \to F_N$ is a surjection onto the infinite set F_N. It follows that F_N is countable and so the same is true for the set of finite subsets of C, $F = \cup_{N=1}^{\infty} F_N$. If $2^C - F$ were countable, then the same would be true of $2^C = (2^C - F) \cup F$, which is false as C is countable. Therefore the (clearly infinite) set of infinite subsets of C, $2^C - F$, must be uncountable.

(b) A surjective map from the natural numbers to a countable set must also be injective.

Solution: FALSE.
Define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} n-1 & \text{if } n \geq 2 \\ 1 & \text{if } n = 1. \end{cases}$$

For any $m \in \mathbb{N}$, $m = f(m+1)$ and so f is surjective. But $f(1) = f(2) = 1$ implies f is not injective.

8 marks 5. Let $\{a_n\}$ be a sequence of real numbers, and let $a \in \mathbb{R}$. Suppose that every subsequence of $\{a_n\}$ has a subsequence convergent to a. Prove that $\{a_n\}$ converges to a.

Solution: Suppose that a_n does not converge to a. Then there is as $\epsilon > 0$ such that for every $N \in \mathbb{N}$, there is an $n > N$ such that $|a_n - a| > \epsilon$. Let $N_1 = 1$, and choose $n_1 > 1$ such that $|a_{n_1} - a| > \epsilon$. Let $N_2 = n_1$, and choose $n_2 > n_1$ such that $|a_{n_2} - a| > \epsilon$. Continuing by induction, we get a subsequence $\{a_{n_k}\}$ such that $|a_{n_k} - a| > \epsilon$ for all k. Clearly, this subsequence has no subsequence convergent to a, which contradicts our assumption.

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