

**Practice Midterm      October 16, 2016      Duration: 50 minutes***This test has 5 questions on 8 pages, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student No.: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	5	Total
Points:	10	10	4	8	8	40
Score:						

**Student Conduct during Examinations**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)?(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. Let  $\{s_n\}$  be a sequence in  $\mathbb{C}$ .

2 marks

(a) Define what it means to say that  $\lim_{n \rightarrow \infty} s_n = s$ .

**Solution:** For every  $\epsilon > 0$  there is an  $N$  such that if  $n \geq N$  then  $|s_n - s| < \epsilon$ .

3 marks

(b) Suppose  $\lim_{n \rightarrow \infty} s_n = s$ . Prove that the sequence  $\{s_n\}$  is bounded.

**Solution:** Take  $\epsilon = 1$  in part (a). Then there is an  $N_1$  such that

$$|s_n| \leq |s| + |s_n - s| \leq |s| + 1 \quad \text{for } n \geq N_1.$$

Let  $M = \max\{|s_1|, \dots, |s_{N_1}|, |s| + 1\}$ . Then  $|s_n| \leq M$  for all  $n \geq 1$ , so the sequence is bounded.

5 marks

(c) Let  $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$ . Suppose  $\lim_{n \rightarrow \infty} s_n = s$ . Prove that  $\lim_{n \rightarrow \infty} \sigma_n = s$ .

**Solution:** By the triangle inequality,

$$|\sigma_n - s| = \frac{1}{n}|s_1 + \cdots + s_n - ns| = \frac{1}{n}|(s_1 - s) + \cdots + (s_n - s)| \leq \frac{1}{n}(|s_1 - s| + \cdots + |s_n - s|).$$

Fix  $\epsilon > 0$ , and choose  $N$  as in part (a). Then  $|s_m - s| < \epsilon$  for  $m \geq N$ , and hence

$$|\sigma_n - s| \leq \frac{1}{n}(|s_1 - s| + \cdots + |s_N - s|) + \frac{n - N}{n}\epsilon$$

for  $n \geq N$ . By part (b), there is an  $M$  such that  $|s_k - s| \leq |s_k| + |s| \leq 2M$  for all  $k$ . Therefore,

$$|\sigma_n - s| \leq \frac{1}{n}2MN + \epsilon \leq \epsilon + \epsilon = 2\epsilon,$$

provided we choose  $n \geq \max(N, MN\epsilon^{-1})$ . This proves that  $\sigma_n \rightarrow s$ .

2 marks

2. (a) Suppose  $E \subset \mathbb{R}$  is nonempty and bounded above. Define precisely what it means to say that  $x = \sup E$ .

**Solution:** It means that (i)  $x$  is an upper bound for  $E$ , i.e., that  $p \leq x$  for all  $p \in E$ , and (ii) if  $y$  is an upper bound for  $E$  then  $x \leq y$ .

8 marks

- (b) Let  $A$  and  $B$  be non-empty subsets of  $[0, \infty)$  which are bounded above, i.e., bounded subsets of non-negative real numbers. Let  $AB = \{xy : x \in A, y \in B\}$ . Prove that  $\sup(AB) = (\sup A)(\sup B)$ .

**Solution:** Let  $a = \sup A$ ,  $b = \sup B$ ,  $c = \sup(AB)$ .

Let  $z \in AB$ . Then  $z = xy$  for some  $x \in A$  and  $y \in B$ . Since  $x \leq a$  and  $y \leq b$ ,  $z \leq ab$ , so  $ab$  is an upper bound for  $AB$  and hence  $c \leq ab$ .

For the reverse inequality note first that if  $a = 0$  or  $b = 0$ , then  $c \geq ab$  is immediate. So assume wlog that  $a, b > 0$ . Let  $0 < \varepsilon < \min(a, b)$ , there exist  $x \in A$  and  $y \in B$  such that  $x > a - \varepsilon$  and  $y > b - \varepsilon$  (since otherwise  $a - \varepsilon$  would be an upper bound for  $A$ , contradicting  $a = \sup A$ , and similarly for  $B$ ). Then we find that  $xy > (a - \varepsilon)(b - \varepsilon) > ab - \varepsilon(a + b)$ , and hence  $c > ab - \varepsilon(a + b)$ . This holds for all  $\varepsilon > 0$ , so we must have  $c \geq ab$ .

4 marks

3. Suppose that  $A, B$  are subsets of a metric space  $X$ . We say that  $A$  is dense in  $B$  iff  $B$  is contained in the closure of  $A$ . Let  $C \subset X$ , and suppose that  $A$  is dense in  $B$ , and that  $B$  is dense in  $C$ . Prove that  $A$  is dense in  $C$ .

**Solution:** We have  $B \subset \bar{A}$  and  $C \subset \bar{B}$ .

Therefore  $C \subset \bar{B} \subset \bar{\bar{A}} = \bar{A}$ , where the last equality holds since  $\bar{A}$  is closed. This proves  $C \subset \bar{A}$ , that is,  $A$  is dense in  $C$ .

8 marks

4. True or False. If True, give a proof; if False, provide a counter-example.

(a) The set of infinite subsets of a countable set is uncountable.

**Solution:** TRUE.

Let  $C$  be a countable set and for each natural number  $N$ , let  $F_N$  be the set of subsets of  $C$  with cardinality at most  $N$ . Define  $\phi : C^N \rightarrow F_N$  by  $\phi(x_1, \dots, x_N) = \{x_i : i \leq N\}$ . Note that this set has cardinality at most  $N$  and so is in  $F_N$ . If  $A \in F_N$ , then for some  $k \leq N$  and distinct  $a_1, \dots, a_k$  in  $C$ ,  $F_N = \{a_1, \dots, a_k\} = \phi(a_1, \dots, a_k, a_k, \dots, a_k) \in \text{Range}(\phi)$ . Hence  $\phi$  is onto.  $C^N$  is countable because  $C$  is, and therefore there is a bijection  $f$  from  $\mathbb{N}$  onto  $C^N$ . Therefore  $\phi \circ f : \mathbb{N} \rightarrow F_N$  is a surjection onto the infinite set  $F_N$ . It follows that  $F_N$  is countable and so the same is true for the set of finite subsets of  $C$ ,  $F = \cup_{N=1}^{\infty} F_N$ . If  $2^C - F$  were countable, then the same would be true of  $2^C = (2^C - F) \cup F$ , which is false as  $C$  is countable. Therefore the (clearly infinite) set of infinite subsets of  $C$ ,  $2^C - F$ , must be uncountable.

(b) A surjective map from the natural numbers to a countable set must also be injective.

**Solution:** FALSE.Define  $f : \mathbb{N} \rightarrow \mathbb{N}$  by

$$f(n) = \begin{cases} n-1 & \text{if } n \geq 2 \\ 1 & \text{if } n = 1. \end{cases}$$

For any  $m \in \mathbb{N}$ ,  $m = f(m+1)$  and so  $f$  is surjective. But  $f(1) = f(2) = 1$  implies  $f$  is not injective.

8 marks
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5. Let  $\{a_n\}$  be a sequence of real numbers, and let  $a \in \mathbb{R}$ . Suppose that every subsequence of  $\{a_n\}$  has a subsequence convergent to  $a$ . Prove that  $\{a_n\}$  converges to  $a$ .

<p><b>Solution:</b> Suppose that <math>a_n</math> does not converge to <math>a</math>. Then there is as <math>\epsilon &gt; 0</math> such that for every <math>N \in \mathbb{N}</math>, there is an <math>n &gt; N</math> such that <math> a_n - a  &gt; \epsilon</math>. Let <math>N_1 = 1</math>, and choose <math>n_1 &gt; 1</math> such that <math> a_{n_1} - a  &gt; \epsilon</math>. Let <math>N_2 = n_1</math>, and choose <math>n_2 &gt; n_1</math> such that <math> a_{n_2} - a  &gt; \epsilon</math>. Continuing by induction, we get a subsequence <math>\{a_{n_k}\}</math> such that <math> a_{n_k} - a  &gt; \epsilon</math> for all <math>k</math>. Clearly, this subsequence has no subsequence convergent to <math>a</math>, which contradicts our assumption.</p>
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