*This midterm has **3 questions** on **6 pages**, for a total of 60 points.*

Duration: 80 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page or on the blank page at the end if you run out of space, **with clear indication on the original page that the solution is continued elsewhere**.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Last name: ____________________ First name: ____________________

Student no.: ____________________________ Course: 421 / 510 (circle one)

Signature: ____

Question:	1	2	3	Total
Points:	24	12	24	60
Score:				

1. Let X be the vector space of all complex-valued sequences $\{a_1, a_2, \dots\}$ such that $\lim_{n\to\infty} a_n$ exists. Define $\|\{a_n\}\| = \sup_n |a_n|$.

12 marks (a) Prove that X is a Banach space with this norm.

Solution:

- If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty}(a_n + b_n) = a + b$ and $\lim_{n\to\infty} \lambda a_n = \lambda a$, hence X is a vector space. (This is optional since the question already refers to X as a vector space and the proof is more or less trivial, but I'm including this part for completeness.)

- Since convergent sequences are bounded, $\|\{a_n\}\|$ is finite for all $\{a_n\} \in X$. We have

$$\|\lambda\{a_n\}\| = \sup_n |\lambda a_n| = |\lambda|\|\{a_n\}\|$$
$$\|\ \{a_n + b_n\}\| = \sup_n |a_n + b_n| \le \sup_n |a_n| + \sup_n |b_n| = \|\{a_n\}\| + \|\{b_n\}\|$$
$$\|\{a_n\}\| = 0 \Leftrightarrow \sup_n |a_n| = 0 \Leftrightarrow a_n = 0\ \forall n,$$

so $\|\cdot\|$ is a norm.

- We have to prove that X is complete. Suppose that $\{x_n\}$ is a Cauchy sequence in X, with $x_n = \{a_{n,j}\}_{j=1}^{\infty}$. Then $\sup_j |a_{n,j} - a_{m,j}| \to 0$ as $m, n \to \infty$. Therefore for each k we have $|a_{n,k} - |a_{m,k}| \le \sup_j |a_{n,j} - a_{m,j}| \to 0$ as $m, n \to \infty$, so that $\{a_{n,k}\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus for every k the limit $\lim_{n\to\infty} a_{n,k} = b_k$ exists.

  Let $y = \{b_k\}_{k=1}^{\infty}$. Using again that $\{x_n\}$ is Cauchy, for any $\epsilon > 0$ there is a N such that $\sup_j |a_{n,j} - a_{m,j}| < \epsilon$ for all $m, n > N$. Taking the limit $m \to \infty$, we get that $\sup_j |a_{n,j} - b_j| < \epsilon$ for all $n > N$. Hence $\|x_n - y\| \to 0$.

  Finally, we have to prove that $y \in X$. Let $a_n = \lim_{j\to\infty} a_{n,j}$, then $|a_n - a_m| = \lim_{k\to\infty} |a_{n,k} - |a_{m,k}| \le \sup_j |a_{n,j} - a_{m,j}| \to 0$ as $m, n \to \infty$, so that $\{a_n\}$ is a Cauchy sequence and therefore has a limit, say $\lim a_n = b$. Let $\epsilon > 0$. Since $a_n \to b$, we can choose N_1 such that $|a_n - b| < \epsilon/3$ for all $n > N_1$. Since $\|x_n - y\| \to 0$, we can also choose N_2 such that $\|\sup_j |a_{n,j} - b_j| < \epsilon/3$ for all $n > N_2$. Fix some $n > \max(N_1, N_2)$. Since $a_{n,j} \to a_n$ as $j \to \infty$, we also have $|a_{n,j} - a_n| < \epsilon/3$ for all $j > N_3$. Then for such j, we have

$$|b_j - b| \le |b_j - a_{n,j}| + |a_{n,j} - a_n| + |a_n - b| < 3\epsilon/3 = \epsilon.$$

  Since $\epsilon > 0$ was arbitrary, we have $b_j \to b$, so that $y \in X$ as required.

(Note: it is possible to make this proof shorter by noticing that X, with this norm, is a subspace of ℓ^∞, and then using that ℓ^∞ is complete. You would still have to prove that X is a closed subspace of ℓ^∞.)

12 marks

(b) Let M be the closed subspace of X consisting of those complex-valued sequences $\{a_1, a_2, \dots\}$ for which $\lim_{n\to\infty} a_n = 0$. (You don't have to prove that this is a closed subspace.) Recall that the quotient space X/M is the space of equivalence classes $x + M$, with $x + M = y + M$ if and only if $x - y \in M$, and with vector operations $(x + M) + (y + M) = (x + y) + M$ and $\lambda(x + M) = \lambda x + M$. The norm on X/M is defined by $\|x + M\| = \inf_{y\in M} \|x + y\|$. Prove that X/M is isometrically isomorphic to $\mathbb{C}$.

Solution: For $x = \{a_n\} \in X$, define $T(x + M) = \lim_{n\to\infty} a_n$.

- This is well defined since the limit exists and if $x, y \in X$, $x = \{a_n\}$, $y = \{b_b\}$, and $x + M = y + M$, then $x - y \in M$, so that $\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n - b_n) = 0$.
- We check that T is linear: if $x, y \in X$, $x = \{a_n\}$, $y = \{b_n\}$, and $\alpha, \beta \in \mathbb{C}$, then $T((\alpha x + \beta y) + M) = \lim_{n\to\infty}(\alpha a_n + \beta b_n) = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n = \alpha T(x + M) + \beta T(y + M)$.
- We check that T is an isometry: let $x \in X$, $x = \{a_n\}$, and let $y \in M$, $y = \{b_n\}$. Then $\|x - y\| = \sup_j |a_j - b_j| \ge \lim_{j\to\infty} |a_j - b_j| = \lim_{j\to\infty} |a_j| = |T(x + M)|$, since $b_j \to 0$. Taking infimum over $y \in M$, we get $\|x + M\| = \inf_{y\in M} \|x - y\| \ge |T(x + M)|$. On the other hand, if we let $z = \{a_n - a\}$ for $a = \lim_{j\to\infty} a_j$, then $z \in M$ and $\|x - z\| = \sup_n |a_n - (a_n - a)| = |a| = |T(x + M)|$, so that $\|x + M\| \le \|x - z\| \le |T(x + M)|$.
- Since T is an isometry, it is one-to-one. To check that T is onto, let $a \in \mathbb{C}$, then $a = T(x + M)$ for $x = \{a, a, \dots\}$ (the constant sequence). Hence T is an isomorphism.

4 marks 2. (a) Give an example of a function $f : \mathbb{R} \to \mathbb{C}$ such that $f \in L^2(\mathbb{R})$ but $f \notin L^3(\mathbb{R})$.

Solution: Let $f(x) = x^{-a}\chi_{(0,1]}$. We have $f(x) \in L^2$ if and only if $2a < 1$, and $f(x) \notin L^3$ if and only if $3a \geq 1$, so it suffices to choose a such that $\frac{1}{3} \leq a < \frac{1}{2}$. For example, we can take $a = 1/3$, so that $f(x) = x^{-1/3}\chi_{(0,1]}$.

4 marks (b) Give an example of a function $f : \mathbb{R} \to \mathbb{C}$ such that $f \in L^3(\mathbb{R})$ but $f \notin L^2(\mathbb{R})$.

Solution: Let $f(x) = x^{-a}\chi_{[1,\infty)}$. We have $f(x) \in L^3$ if and only if $3a > 1$, and $f(x) \notin L^2$ if and only if $2a \leq 1$, and so it suffices to choose a such that $\frac{1}{3} < a \leq \frac{1}{2}$. For example, we can take $a = 1/2$, so that $f(x) = x^{-1/2}\chi_{[1,\infty)}$.

4 marks (c) Is there a function $f : [0,1] \to \mathbb{C}$ such that $f \in L^3([0,1])$ but $f \notin L^2([0,1])$? If yes, give an example. If not, explain why.

Solution: No. Applying Hölder's inequality with exponents $p = 3/2$ and $q = 3$, we get

$$\int_0^1 |f|^2 = \int_0^1 |f|^2\chi_{[0,1]} \leq \left(\int_0^1 (|f|^2)^{3/2}\right)^{2/3} \left(\int \chi_{[0,1]}^3\right)^{1/3} = \left(\int_0^1 |f|^3\right)^{2/3},$$

so that if $f \in L^3([0,1])$ then also $f \notin L^2([0,1])$. (It would also suffice to refer to L^p inclusion for finite measures.)

3. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Let p, q, r be exponents such that $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

12 marks

(a) For $g \in L^q(\mu)$, define $T_g(f) = fg$. Prove that T_g is a bounded linear operator from $L^p(\mu)$ to $L^r(\mu)$.

Solution: For $a, b \in \mathbb{C}$ and $f_1, f_2 \in L^p$, we have $T_g(af_1 + bf_2) = (af_1 + bf_2)g = af_1 g + bf_2 g = aT_g f_1 + bT_g f_2$, so that T_g is linear. Since $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have $p \ge r$ and $q \ge r$.

- If both p and q are finite, then $\frac{r}{p} + \frac{r}{q} = 1$ and we can apply Hölder's inequality with exponents p/r and q/r:

$$\int |fg|^r = \int |f|^r |g|^r \le \Big(\int (|f|^r)^{p/r} \Big)^{r/p} \Big(\int (|g|^r)^{q/r} \Big)^{r/q}$$

$$= \Big(\int |f|^p \Big)^{r/p} \Big(\int |g|^q \Big)^{r/q} = \|f\|_p^r \|g\|_q^r.$$

  Hence $\|T_g f\|_r = \|fg\|_r \le \|f\|_p \|g\|_q$. Therefore $T_g f \in L^r$ and $T_g \in \mathcal{L}(L^p, L^r)$, with norm bounded by $\|g\|_q$.

- If $p = \infty$, then $q = r$ and

$$\|fg\|_r \le \|f\|_\infty \|g\|_r =\le \|f\|_\infty \|g\|_q$$

  so $T_g \in \mathcal{L}(L^\infty, L^r)$, with $\|T_g\| \le \|g\|_q$. Similarly, if $q = \infty$, we have $p = r$ and $\|fg\|_r \le \|f\|_r \|g\|_\infty =\le \|f\|_p \|g\|_\infty$, so that $T_g \in \mathcal{L}(L^p, L^p)$, with $\|T_g\| \le \|g\|_\infty$.

Alternatively, to prove that T_g is bounded, it suffices to refer to a version of Hölder's inequality (often called "generalized Hölder's inequality") saying that $\|fg\|_r \le \|f\|_p \|g\|_q$ with exponents as above. The argument above provides a proof of that generalized inequality.

12 marks

(b) Let $g : \Omega \to \mathbb{C}$ be measurable but not in $L^\infty(\mu)$. Define $T_g f = fg$ as in (a). Prove that if either $p = \infty$, or if $1 \le p < \infty$ and μ is semifinite, then T_g cannot be a bounded operator from $L^p(\mu)$ to $L^p(\mu)$. (Semifinite means that for every measurable set $A \subset \Omega$ with $\mu(A) = \infty$, there is a measurable set $B \subset A$ with $0 < \mu(B) < \infty$.)

Solution: Assume that $g \notin L^\infty$. This means that for every $n \in \mathbb{N}$, the set $A_n = \{x : |g(x)| > n\}$ has positive measure.

- Assume first that μ is semifinite and $1 \le p < \infty$. Choose $B_n \subset A_n$ so that $0 < \mu(B_n) < \infty$. Let $f_n = \chi_{B_n}$, then $\|f_n\|_p = \mu(B_n)^{1/p}$ and $\|f_n g\|_p \ge (\int_{B_n} n^p)^{1/p} = n\mu(B_n)^{1/p} = n\|f_n\|_p$, so that T_g cannot be bounded on L^p.
- If $p = \infty$, let $f_n = \chi_{A_n}$, then $\|f_n\|_\infty = 1$ and $\|f_n g\|_\infty \ge n$, so again T_g cannot be bounded on L^∞.