*This midterm has **5 questions** on **6 pages**, for a total of 60 points.*

Duration: 80 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page or on the blank page at the end if you run out of space, **with clear indication on the original page that the solution is continued elsewhere**.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Last name: ____________________ First name: ____________________

Student no.: ____________________________ Course: 421 / 510 (circle one)

Signature: ____

Question:	1	2	3	4	5	Total
Points:	16	16	8	10	10	60
Score:						

1. Let $f_n = \chi_{[2^n, 2^{n+1}]}$ for $n \in \mathbb{N}$.

8 marks (a) Let $1 < p < \infty$. Is it true that $f_n \to 0$ weakly in $L^p(\mathbb{R})$ as $n \to \infty$? Explain why or why not.

Solution:
No. We have $\|f_n\|_p = (\int_{2^n}^{2^{n+1}} 1)^{1/p} = 2^{n/p} \to \infty$ as $n \to \infty$. But if the sequence $\{f_n\}$ were weakly convergent (to 0 or anything else), then by the Uniform Boundedness Principle it would have to be bounded.

8 marks (b) Is it true that $f_n \to 0$ weak* in $L^\infty(\mathbb{R}) = (L^1(\mathbb{R}))^*$ (i.e. as linear functionals on L^1) as $n \to \infty$? Explain why or why not.

Solution:
Yes. If $g \to L^1$, then $\sum_1^\infty \int |f_n g| = \int_2^\infty |g| < \|g\|_1 < \infty$. Hence $|\int f_n g| \leq \int |f_n g| \to 0$ as $n \to \infty$.

8 marks

2. (a) Let X, Y be normed vector spaces, and let $T \in \mathcal{L}(X, Y)$. Prove that if $\{x_n\}$ is a sequence in X such that $x_n \to x$ weakly for some $x \in X$, then $Tx_n \to Tx$ weakly in Y.

Solution: Suppose that $\{x_n\} \subset X$ and $x_n \to x$ weakly. Let $f \in Y^*$, then $f(Tx_n) = (T^t f)(x_n) \to (T^t f)(x) = f(Tx)$. Therefore $Tx_n \to Tx$ weakly.

8 marks

(b) Let X be a normed vector space, and let $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$ be sequences such that $x_n \to x$ weakly in X and $\|f_n - f\| \to 0$. Prove that $f_n(x_n) \to f(x)$.

Solution:

Let f_n, x_n be as above, then by the Uniform Boundedness Principle there is a constant C such that $\|x_n\| \leq C$ for all n. Then

$$\begin{aligned} |f_n(x_n)| - f(x)| &\leq |(f_n - f)(x_n)| + |f(x_n) - f(x)| \\ &\leq \|f_n - f\| \, \|x_n\| + |f(x_n - x)| \\ &\leq C\|f_n - f\| + |f(x_n - x)|. \end{aligned}$$

As $n \to \infty$, both terms go to 0 as required.

8 marks 3. Let X, Y be normed vector spaces, and let $T \in \mathcal{L}(X, Y)$. Prove that if T is invertible, then T^t is also invertible and $(T^t)^{-1} = (T^{-1})^t$.

Solution: If $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$, then $(ST)^t = T^t S^t$. To prove this, we check that for all $x \in X$, $f \in Z^*$, we get

$$\Big((ST)^t f\Big)(x) = f(STx) = (S^t f)(Tx) = \Big(T^t S^t f\Big)(x).$$

Applying this with $S = T^{-1} \in \mathcal{L}(Y, X)$, we get $T^t S^t = (ST)^t = I_X^t = I_{Y^*}$ and similarly, $S^t T^t = (TS)^t = I_Y^t = I_{X^*}$. Therefore $S^t = (T^t)^{-1}$.

Since $S = T^{-1}$ is bounded and $\|S^t\| = \|S\|$, S^t is bounded, Hence T^t is invertible.

(More direct proofs are also possible. Note however that to prove that $S^t = (T^t)^{-1}$, we have to prove both $T^t S^t = I_{Y^*}$ and $S^t T^t = I_{X^*}$. It would not suffice to prove only one of them.)

10 marks 4. Let X be a normed vector space. Let $\{f_n\} \subset X^*$ be a sequence of functionals such that

- $\|f_n\| \le C$ for some constant $C > 0$ and all $n \in \mathbb{N}$,
- there is a $f \in X^*$ such that $f_n(x) \to f(x)$ for all $x \in S$, where S is a dense subset of X.

Prove that f_n converge weak* to f.

Solution: Let $x \in X$; we need to prove that $f_n(x) \to f(x)$. Let $\epsilon > 0$, and choose $y \in S$ such that $\|x - y\| < \epsilon$. Then

$$\begin{aligned} |f_n(x) - f(x)| &\le |f_n(x) - f_n(y)| + |f_n(y) - f(y)| + |f(y) - f(x)| \\ &\le \|f_n\| \, \|x - y\| + |f_n(y) - f(y)| + \|f\| \, \|x - y\| \\ &\le C\epsilon + |f_n(y) - f(y)| + \|f\|\epsilon. \end{aligned}$$

If $n > N$ for some N large enough, we have $|f_n(y) - f(y)| < \epsilon$ by the assumed convergence on S. Therefore $|f_n(x) - f(x)| < (C + \|f\| + 1)\epsilon$ for all $n > N$. Since $\epsilon > 0$ was arbitrary, we have $f_n(x) - f(x) \to 0$, as required.

10 marks 5. Recall definitions from class: if X is a normed vector space, and it M, N are closed linear subspaces of X and X^* respectively, then $M^0 = \{f \in X^* : f(x) = 0 \text{ for all } x \in M\}$ and $N^\perp = \{x \in X : f(x) = 0 \text{ for all } f \in N\}$.

Prove that if M is a closed subspace of X, then $(M^0)^\perp = M$. (This was done in class. You are being asked to reproduce the proof here, not just state the relevant result.)

Solution:

- If $x \in M$, then $f(x) = 0$ for all $f \in M^0$ by definition, so $M \subset (M^0)^\perp$.
- If $x \notin M$, then by Hahn-Banach we can find a functional $f \in X^*$ such that $f(x) \neq 0$ but $f \equiv 0$ on M. Then $f \in M^0$, but $f(x) \neq 0$, so that $x \notin (M^0)^\perp$. Hence $(M^0)^\perp \subset M$.