MATH 541 HOMEWORK 1

Due Friday, October 7, 2005

1. Let $\phi \in \mathcal{S}$, $\int \phi = 1$, and let $\phi^\epsilon = \epsilon^{-n}\phi(x/\epsilon)$. Prove the following (see Lemma 3.2, pp. 16-17, in the textbook):

   (a) If $f : \mathbb{R}^n \to \mathbb{C}$ is continuous and $|f| \to 0$ at ∞, then $\phi^\epsilon * f \to f$ uniformly as $\epsilon \to 0$.

   (b) If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then $\phi^\epsilon * f \to f$ in L^p as $\epsilon \to 0$.

   Does (b) remain true when $p = \infty$? Explain why or why not.

2. This is another manifestation of the general principle that the decay of $\hat{f}$ is related to the regularity of f.

   (a) Suppose that $f, \hat{f} \in L^1(\mathbb{R})$, and that

   $$|\hat{f}(\xi)| \le C|\xi|^{-1-\alpha} \text{ for some } C > 0, 0 < \alpha < 1. \tag{1}$$

   Prove that f satisfies the Hölder condition

   $$|f(x+h) - f(x)| \le M|h|^\alpha,\ \ x, h \in \mathbb{R}, \tag{2}$$

   for some constant M. (Hint: use the inversion formula.)

   (b) What exponent should replace $-1-\alpha$ in (1) for the conclusion (2) to hold in $\mathbb{R}^n$?

3. Let $f \in L^2(\mathbb{R})$, $\int |f|^2 = 1$. Prove that

   $$\Big(\int x^2 |f(x)|^2 dx\Big)\Big(\int \xi^2 |\hat{f}(\xi)|^2 d\xi\Big) \ge (16\pi^2)^{-1}.$$

   (Hint: integrate $\int |f|^2 = \int |f|^2 \cdot 1$ by parts and use Cauchy-Schwarz.) This is known as the *uncertainty principle*: in quantum mechanics, the two integrals are interpreted as "uncertainty of position" and "uncertainty of momentum", respectively, of the quantum particle.