MATH 541 HOMEWORK 2

Due Friday, November 18, 2005

1. **Riemann-Lebesgue Lemma:** Prove that if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

2. **Poisson Summation Formula:** Let $f \in \mathcal{S}(\mathbb{R}^n)$, then we define the *periodization* of f:

$$F_1(x) = \sum_{\nu \in \mathbb{Z}^n} f(x + \nu).$$

We also let

$$F_2(x) = \sum_{\nu \in \mathbb{Z}^n} \widehat{f}(\nu) e^{2\pi i \nu \cdot x}.$$

(a) Prove that the sums defining F_1 and F_2 converge absolutely and uniformly, and that F_1, F_2 are periodic in the sense that $F_i(x+\nu) = F_i(x)$ for all $\nu \in \mathbb{Z}^n$, $i = 1, 2$.

(b) Prove that $F_1 = F_2$, by verifying that F_1 and F_2 have the same Fourier series. In particular, $F_1(0) = F_2(0)$, i.e.

$$\sum_{\nu \in \mathbb{Z}^n} f(\nu) = \sum_{\nu \in \mathbb{Z}^n} \widehat{f}(\nu).$$

This is known as the *Poisson summation formula.*

(c) Prove that if $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}.$$

(Hint: apply (b) to the function $g(x) = \max(1 - |x|, 0)$.)

(d) Prove that if $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = \frac{\pi}{\tan(\pi\alpha)}.$$

(Hint: Prove this first for $0 < \alpha < 1$, by integrating the formula above.)

3. **Fourier transform of the Hausdorff measure on a Cantor set:** Let $C \subset [-1/2, 1/2]$ be the middle-thirds Cantor set, so that $\dim(C) = \frac{\log 2}{\log 3}$, and let μ be the restriction of $H_{\log 2/log3}$ to C, i.e. $\mu(E) = H_{\log 2/log3}(E \cap C)$. Then $\mu \in P(C)$. Prove that

$$\widehat{\mu}(x) = \prod_{k=1}^{\infty} \cos(3^{-k}x). \tag{1}$$

One way to do this is outlined below, but you should feel free to look for alternative solutions.

(a) Prove that the partial products $\prod_1^N$ in (1) converge uniformly on compact intervals, hence the right side of (1) is a continuous function.

(b) Prove that μ satisfies the equation

$$\int f d\mu = \frac{1}{2} \int (f(3x+1)\chi_{[-\frac{1}{2},-\frac{1}{6}]}(x) + f(3x-1)\chi_{[\frac{1}{6},\frac{1}{2}]}(x))d\mu$$

for all $f \in C[-1/2, 1/2]$. Deduce that

$$\widehat{\mu}(x) = \cos(\frac{x}{3})\widehat{\mu}(\frac{x}{3}). \tag{2}$$

(c) Prove that a non-zero continuous function satisfying (2) and equal to 1 at 0 must be identical to the right side of (1).