MATHEMATICS 541, PROBLEM SET 1
Due on Wednesday, September 22

Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

1. (Weierstrass approximation theorem) Let f be a continuous function on a closed and bounded interval $[a,b]$, with values in $\mathbb{R}$. Prove that f can be approximated by polynomials uniformly on $[a,b]$: for every $\epsilon > 0$ there is a polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a,b]$. (Approximate f by trigonometric polynomials first, then approximate the trigonometric polynomials by polynomials.)

2. (Fejér's lemma) Prove that if $f \in L^1(\mathbb{T})$ and $g \in L^\infty(\mathbb{T})$, then
$$\lim_{n\to\infty} \int_0^1 f(x) g(nx) dx = \widehat{f}(0)\widehat{g}(0).$$

3. (Stein and Shakarchi, page 66) Let $D_n(x) = \sum_{n=-N}^{N} e^{2\pi i n x} = \dfrac{\sin((2N+1)\pi x)}{\sin(\pi x)}$.

   Prove that
$$\int_0^1 |D_N(x)| dx \geq c \log N$$
   for some $c > 0$. (Hint: Prove that $|D_N(x)| \geq c \dfrac{\sin((2N+1)\pi x)}{|x|}$, hence by changing variables
$$\int_0^1 |D_N(x)| dx \geq c \int_\pi^{N\pi} \frac{|\sin t|}{|t|} dt - C.$$
   Split up the integral into integrals over $[k\pi, (k+1)\pi]$, $k = 1, 2, \dots,$ and use that $\sum_{k=1}^{N} (1/k) \geq c \log N$.)

4. Prove that for each $n \geq 1$ there is a continuous function f_n on $\mathbb{T}$ such that $|f_n| \leq 1$, but $|S_n(f_n)(0)| \geq c' \log n$, where S_n is the partial Fourier

sum $S_n(f) = \sum_{k=-n}^{n} \widehat{f}(n)e^{2\pi inx} = f * D_n$. (Hint: Let

$$g_n(x) = \operatorname{sgn} D_n(x) = \begin{cases} 1 & \text{if } D_n(x) > 0, \\ 0 & \text{if } D_n(x) = 0, \\ -1 & \text{if } D_n(x) < 0. \end{cases}$$

Check that g_n has the desired property except for the continuity. To fix that, approximate g_n by continuous functions.)