MATHEMATICS 541, PROBLEM SET 2
Due on Friday, October 8

Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

1. (Stein-Shakarchi, pp. 65-66)

   (a) You may have seen the proof that
   $$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$
   using complex analysis. Here you are asked to prove it using harmonic analysis: check that the function $\dfrac{1}{\sin(\pi x)} - \dfrac{1}{\pi x}$ is continuous on $[0, 1]$, apply the Riemann-Lebesgue lemma, and use that $\int_0^1 D_N(t)dt = 1$.

   (b) In the construction of a function with divergent Fourier series, we used that
   $$f(x) = \sum_{0<|n|<N} \frac{e^{2\pi inx}}{n}$$
   is bounded uniformly for all $x \in \mathbb{T}$ and $N \in \mathbb{N}$. Prove this. (Hint: prove first that $f(x) = 2\pi i \displaystyle\int_0^x (D_N(t) - 1)dt$, then use part (a).)

2. (Decay of Fourier coefficients.) Let $k \geq 1$ be an integer.

   (a) Prove that if $f : \mathbb{T} \to \mathbb{C}$ is k times differentiable and $f^{(k-1)}$ is absolutely continuous, then $|\widehat{f}(n)| = o(n^{-k})$ as $n \to \infty$.

   (b) (Katznelson, p.28) Prove that if f is a function $\mathbb{T} \to \mathbb{C}$ such that
   $$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|\, |n|^{\ell} < \infty,$$
   then f is ℓ-times continuously differentiable. Deduce that if $\widehat{f}(n) = O(|n|^{-k})$ for some $k > 2$, then f is $(k-2)$-times continuously differentiable.

3. (Stein-Shakarchi, page 123) Prove that for any $a \neq 0$, and σ with $0 < \sigma < 1$, the sequence $\{an^\sigma\},\ n = 1, 2, \ldots,$ is equidistributed on $\mathbb{T}$. (Hint: Prove that for all $b \neq 0$,

$$\sum_{n=1}^{N} e^{2\pi i b n^\sigma} - \int_1^N e^{2\pi i b n^\sigma} = O\Big(\sum_{n=1}^{N} n^{-1+\sigma}\Big);$$

deduce that $\sum_{n=1}^{N} e^{2\pi i b n^\sigma} = O(N^\sigma) + O(N^{1-\sigma})$.)