MATHEMATICS 541, PROBLEM SET 3
Due on Friday, October 29

Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

1. (Decay of $\widehat{f}$ implies regularity of f; Stein-Shakarchi, p. 162) Suppose that $f, \widehat{f} \in L^1(\mathbb{R})$ and for some $0 < \alpha < 1$ we have

$$|\widehat{f}(\xi)| \leq C|\xi|^{-1-\alpha} \text{ for some } C > 0.$$

Prove that f satisfies the uniform Hölder condition with exponent α: there is a constant $M > 0$ such that

$$|f(x+h) - f(x)| \leq M|h|^\alpha, \; x, h \in \mathbb{R}.$$

(Hint: use the inversion formula to estimate the left side by an integral involving $\widehat{f}$.)

2. (A variant of the uncertainty principle; Stein-Shakarchi, p. 168) Suppose that $f \in \mathcal{S}(\mathbb{R})$, and that $f, \widehat{f}$ are essentially localized on intervals $[-R_1, R_1]$ and $[-R_2, R_2]$ respectively, i.e.

$$\int_{-R_1}^{R_1} x^2 |f(x)|^2 dx \geq \frac{1}{2} \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx,$$

$$\int_{-R_2}^{R_2} \xi^2 |\widehat{f}(\xi)|^2 d\xi \geq \frac{1}{2} \int_{-\infty}^{\infty} |\xi^2 |\widehat{f}(\xi)|^2 d\xi.$$

Prove that there is a constant $c > 0$ (independent of f) such that $R_1 R_2 \geq c$.

3. Prove the following form of Young's inequality (more general than what we proved in class): if $p, q, r \in [1, \infty)$ are exponents such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, then for any $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ we have that $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_q.$$

4. Everyone needs a break: you get 10 marks for free.