MATHEMATICS 541, PROBLEM SET 4
Due on Friday, November 19

Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

1. (Stein-Shakarchi, p. 165) Recall the Poisson Summation Formula: if $f \in \mathcal{S}(\mathbb{R})$, then
$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n). \tag{1}$$
   (a) Prove that if $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, then
$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}. \tag{2}$$
   (Hint: Apply (1) to the function $g(x) = \max(1-|x|, 0)$. Note that g is not in $\mathcal{S}$, so you need to justify the application of (1).)

   (b) Prove that if $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, then
$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = \frac{\pi}{\tan(\pi\alpha)}.$$
   (Hint: Prove this first for $0 < \alpha < 1$, by integrating (2).)

2. In this question, all integrals over $\mathbb{R}$ are understood as limits $\lim_{R\to\infty} \int_{-R}^{R}$.

   (a) Prove that $\int_{-\infty}^{\infty} e^{-\pi i \lambda x^4}\, dx = O(\lambda^{-1/4})$ as $\lambda \to \infty$. (Hint: use a change of variables, and recall the formula for the Fourier transform of $f(x) = |x|^{-\alpha}$.)

   (b) Let
$$I(\lambda) = \int_{-\infty}^{\infty} a(x) e^{-\pi i \lambda x^4}\, dx,$$
   where $a \in C_c^\infty(\mathbb{R})$ is real-valued and $a \equiv 1$ on some interval $(-\epsilon, \epsilon)$ with $\epsilon > 0$. Prove that $|I(\lambda)| \le C|\lambda|^{-1/4}$ as $\lambda \to \infty$, by writing $a(x) = 1 + (a(x) - 1)$, splitting up the integral accordingly, and estimating the second integral by parts.

(The assumption that $a \equiv 1$ near 0 can be dropped with a little more work. If x^4 is replaced by x^k, the estimate is $O(\lambda^{-k})$ instead.)