MATHEMATICS 541, PROBLEM SET 5
Due on Friday, December 3

Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

1. Prove that if $f \in \mathcal{S}(\mathbb{R})$ and $\int_{-\infty}^{\infty} f(x)dx = 0$, then $Hf \in L^1(\mathbb{R})$.

2. Here is an alternative proof of L^p boundedness of the Hilbert transform. The steps below outline the proof that H is bounded on $L^p(\mathbb{R})$ if $p = 2^k$, $k = 2, 3, \dots$. As in class, this implies by duality and interpolation that H is bounded on $L^p(\mathbb{R})$ for all $p \in (1, \infty)$.

   (a) Prove that for all real-valued $f \in \mathcal{S}$ we have

   $$(Hf)^2 = f^2 + 2H(f \cdot Hf). \tag{1}$$

   There are several ways to do this. One is as follows: let $f \in \mathcal{S}$ and $g = Hf$, then $fg \in \mathcal{S}$ (prove it!). We have seen that there is a function $F(z)$, analytic in the upper half-plane, such that $F(x + iy) \to f(x) + ig(x)$ as $y \to 0$. Then the function $-iF^2(z)$ is also analytic in the upper half-plane and its limit as $y \to 0$ is $2fg - i(f^2 - g^2)$. What does this say about the Hilbert transform of $2fg$?

   (b) Deduce from (1) that if $\|Hf\|_p \le C_p \|f\|_p$, then

   $$\|Hf\|_{2p} \le (C_p + \sqrt{C_p + 1})\|f\|_{2p}.$$

3. End of term: 10 marks for free!