MATHEMATICS 542, PROBLEM SET 2
Due on Monday, November 24, 2008

1. Find all functions $f : \mathbb{Z}_N \to \mathbb{C}$ such that $|f(x)| \leq 1$ for all x and
   (a) $\|f\|_{U^2} = 1$,
   (b) $\|f\|_{U^3} = 1$.

2. Prove that $\|f\|_{U^2} \leq \|f\|_{U^3} \leq \|f\|_{U^4} \leq \dots$ for all $f : \mathbb{Z}_N \to \mathbb{C}$.

3. Prove that for every $f : \mathbb{Z}_N \to \mathbb{C}$ we have $\|f\|_{U^3} = \sqrt{N}\|\widehat{f}\|_{U^3}$.

4. Let $A, B, C \subset Z$, where Z is an abelian group, and let $G \subset A \times B$, $H \subset B \times C$ be such that $|G| \geq (1-\epsilon)|A||B|$ and $|H| \geq (1-\epsilon)|B||C|$ for some $\epsilon \in (0, 1/4)$. Prove that there are subsets $A' \subset A$ and $C' \subset C$ such that $|A'| \geq (1-\sqrt{\epsilon})|A|$, $|B'| \geq (1-\sqrt{\epsilon})|B|$, and
$$|A' - C'| \leq \frac{ST}{(1-2\sqrt{\epsilon})|B|},$$
where $S = |\{a - b : \ (a,b) \in G\}|, \ T = |\{b - c : \ (b,c) \in H\}|$.
(Hint: Prove that there are at most $\sqrt{\epsilon}|B|$ elements b of B such that
$$|\{a \in A : \ (a,b) \in G\}| \leq (1-\sqrt{\epsilon})|A|,$$
and similarly with A, B, G replaced by B, C, H.)

5. Recall the following version of Roth's theorem: if $f : \mathbb{Z}_N \to [0, M]$ is a function such that $\sum_{x \in \mathbb{Z}_N} f(x) = \delta N$, then
$$\sum_{x,r} f(x)f(x+r)f(x+2r) \geq c(M,\delta)N^2.$$
Use an example from an earlier part of the course to show that one cannot take $c(M,\delta) = M^{-k}c'(\delta)$ for any k independent of M, δ.

6. (Bonus problem) Let $f : \mathbb{Z}_N \to \mathbb{C}$, $|f(x)| \leq 1$ for all x, and let $f^{(r)}(x) = f(x)\overline{f(x+r)}$. Suppose that $G = \{r : \|f^{(r)}\|_{U^2} = 1\}$ has cardinality greater than cN for some c close to 1. Must f be as in 1(b)? If yes, how close to 1 does c need to be?