MATHEMATICS 542, PROBLEM SET 1
Due on Wednesday, January 26.

Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

For full credit, you need to solve 3 problems. If you solve all 4, the lowest problem score will be dropped.

1. (Stein-Shakarchi, p. 146) Let
$$f(x) = \begin{cases} \frac{1}{|x|(\log 1/|x|)^2} & \text{if } |x| \le 1/2, \\ 0 & \text{if } |x| > 1/2. \end{cases}$$
   (a) Check that f is integrable.
   (b) Prove that for $|x| \le 1/2$,
$$Mf(x) \ge \frac{c}{|x|(\log 1/|x|)}$$
   with some $c > 0$. Conclude that Mf is not locally integrable.

2. Use the Lebesgue density theorem to prove the following: if a set $E \subset \mathbb{R}$ has positive measure, then there is an $x \in \mathbb{R}$ and $r > 0$ such that $x, x+r, x+2r$ are all in E. (Hint: look for $x, x+r, x+2r$ all close to a density point.)

3. Let $\phi(x) = h(|x|)$ for $x \in \mathbb{R}^d$, where $h : (0, \infty) \to (0, \infty)$ is positive and decreasing. Let $\phi_\epsilon(x) = \epsilon^{-d}\phi(x/\epsilon)$. Prove that if $\phi \in L^1(\mathbb{R}^d)$, then
$$\sup_{\epsilon > 0} |\phi_\epsilon * f(x) \le \|\phi\|_1 Mf(x), \ \ f \in L^1(\mathbb{R}^d). \tag{1}$$
   (Hint: Approximate ϕ from below by simple functions of the form $\sum c_j \mathbf{1}_{B_j}$, where B_j are balls in $\mathbb{R}^d$.)

4. Define ϕ_ϵ as in Question 3. Prove that for all $f \in L^1(\mathbb{R}^d)$ we have
$$\lim_{\epsilon \to 0} \phi_\epsilon * f(x) = \|\phi\|_1 f(x), \ \text{a.e. } x.$$
   (Hint: Prove this first for continuous functions f. To pass from continuous to L^1 functions, combine (1) with the same argument as in the proof of the Lebesgue differentiation theorem.)