MATHEMATICS 542, PROBLEM SET 3
Due on Friday, March 11.

Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

For full credit, you need to solve 3 problems. If you solve all 4, the lowest problem score will be dropped.

1. Recall that the maximal Radon transform of a function $f \in C_c(\mathbb{R}^d)$ is defined by

$$R^*f(\omega) = \sup_{t\in\mathbb{R}} \int_{P_{t,\omega}} |f|dy, \ \omega \in S^{d-1},$$

where $P_{t,\omega} = \{x \in \mathbb{R}^d : \ x \cdot \omega = t\}$ and the integration is with respect to the $(d-1)$-dimensional measure. We proved that for f as above,

$$\int_{S^{d-1}} R^*f(\omega)d\sigma(\omega) \le C(\|f\|_1 + \|f\|_2).$$

Prove that a similar inequality without the L^2 norm,

$$\int_{S^{d-1}} R^*f(\omega)d\sigma(\omega) \le C\|f\|_1 \qquad (1)$$

is false. (Hint: the function $f(x) = |x|^{-d+\delta}$ with $\delta \in (0,1)$ provides a counterexample to (1) but is neither continuous not compactly supported. Use it to construct a family of functions in $C_c(\mathbb{R}^d)$ for which (1) fails.)

2. The (upper) Minkowski dimension of a compact set $E \subset \mathbb{R}^d$ is defined as

$$\dim_M(E) = \inf\{\beta \ge 0 : \ |E_\delta| \le C_\beta \delta^{d-\beta} \text{ for some } C_\beta \text{ and all } \delta > 0.\}$$

Prove that $\dim(E) \le \dim_M(E)$ for any compact $E \subset \mathbb{R}^d$, where $\dim(E)$ is the Hausdorff dimension of E.

3. (a) Prove that any countable set has Hausdorff dimension 0.

(b) Find the upper Minkowski dimension of the set

$$\{0, 1, 1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{4}, \dots\}.$$

4. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$, and let μ be a nonnegative, finite, compactly supported Borel measure on $\mathbb{R}^d$. Prove that

$$\int \widehat{f}(x)\overline{\widehat{g}(x)}d\mu(x) = \int f(x)(\overline{g} * \widehat{\mu})(x)dx.$$

(We used this identity in the proof of the Tomas-Stein theorem.)