MATHEMATICS 542, PROBLEM SET 4
Due on Friday, April 11.

Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

For full credit, you need to solve 3 problems. If you solve all 4, the lowest problem score will be dropped.

In all questions below, $\dim E$ will denote the Hausdorff dimension of a Borel set E.

1. Let $E \subset \mathbb{R}^m$, and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz mapping: $|f(x) - f(y)| \leq C|x - y|$. Prove that $\dim f(E) \leq \dim E$.

2. Let $E \subset \mathbb{R}^m$ and $F \subset \mathbb{R}^n$ be compact sets. Prove that
$$\dim(E \times F) \geq \dim(E) + \dim(F).$$
(Hint: use Frostman's lemma.)

3. (Stein-Shakarchi) Prove that the inequality in Question 2 can be strict, by constructing compact sets $U, V \subset \mathbb{R}$ such that
$$\dim(U) = \dim(V) = 0, \ \dim(U \times V) \geq 1.$$
Hint: let A_j, B_j be two sequences of positive integers such that $A_1 < B_1 < A_2 < \cdots < A_j < B_j < A_{j+1} < \ldots$. Assume that
$$\lim_{j\to\infty} B_j/A_j = \lim_{j\to\infty} A_{j+1}/B_j = \infty.$$
For $j = 1, 2, \ldots$, let
$$I_j = \{A_j, A_j + 1, \ldots, B_j\}, \ J_j = \{B_j, B_j + 1, \ldots, A_{j+1} - 1\}.$$
For any $x \in [0, 1]$, let $x = .x_1x_2\ldots$ be the expansion of x in base 2. Let
$$U = \{x \in [0, 1] : \ x_n = 0 \text{ for all } n \in \bigcup_j I_j\},$$
$$V = \{x \in [0, 1] : \ x_n = 0 \text{ for all } n \in \bigcup_j J_j\}.$$
Prove that U and V have the desired properties.

4. For a compact set $E \subset \mathbb{R}^d$ and $s > 0$, define the *s-capacity* of E:

$$\mathcal{C}_s(E) = \sup\{I_s(\mu)^{-1} : \ \mu \in P(E)\}.$$

Prove that $\mathcal{C}_s(E) > 0$ if and only if there is a $\mu \in P(E)$ such that the function $V_\mu(x) = \int |x - y|^{-s} d\mu(y)$ is bounded on $\mathbb{R}^d$.