Chapter 4

Applications of the definite integral to rates, velocities and densities

4.1 Displacement, velocity and acceleration

Recall from our study of derivatives that for $x(t)$ the position of some particle at time t, $v(t)$ its velocity, and $a(t)$ the acceleration, the following relationships hold:

$$\frac{dx}{dt} = v,$$

$$\frac{dv}{dt} = a.$$

(Velocity is the derivative of position and acceleration is the derivative of velocity.) This means that position is an anti-derivative of velocity and velocity is an anti-derivative of acceleration.

Since position, $x(t)$, is an anti-derivative of velocity, $v(t)$, by the Fundamental Theorem of Calculus, it follows that over the time interval $T_1 \leq t \leq T_2$,

$$\int_{T_1}^{T_2} v(t)\ dt = x(t)\Big|_{T_1}^{T_2} = x(T_2) - x(T_1). \tag{4.1}$$

Similarly, since velocity is an anti-derivative of acceleration, the Fundamental Theorem of Calculus says that

$$\int_{T_1}^{T_2} a(t)\ dt = v(t)\Big|_{T_1}^{T_2} = v(T_2) - v(T_1). \tag{4.2}$$

We now consider two types of motion, and use these results to derive some well-known results for the motion of particles under uniform velocity or uniform acceleration situations. Before doing so, we define the concept of displacement.

Displacement

The *displacement*, $x(T_2)-x(T_1)$, of a particle, over the time interval $T_1 \leq t \leq T_2$ is the net difference between its final and initial positions. For example, if you drive from home to work in the morning, and drive back at night, then your displacement over that day is zero. (You may have driven a long distance in getting there and back.)

Suppose we label the initial time zero ($T_1 = 0$) and the final time T (i.e. $T_2 = T$) then the definite integral of the velocity,

$$\int_0^T v(t)\ dt = x(T) - x(0),$$

is the displacement over the time interval $0 \leq t \leq T$.

We can similarly consider the net change prescribed by equation 4.2 over this time interval:

$$\int_0^T a(t)\ dt = v(t)\Big|_0^T = v(T) - v(0)$$

is the net change in velocity between time 0 and time T, (though this quantity does not have special terminology).

Geometric interpretations

Suppose we are given a graph of the velocity $v(t)$, as shown on the left of Figure 4.1. Then by the definition of the definite integral, we can interpret $\int_{T_1}^{T_2} v(t)\ dt$ as the "area" associated with the curve (counting positive and negative contributions) between the endpoints T_1 and T_2. Then according to the above observations, this area represents the displacement of the particle between the two times T_1 and T_2.

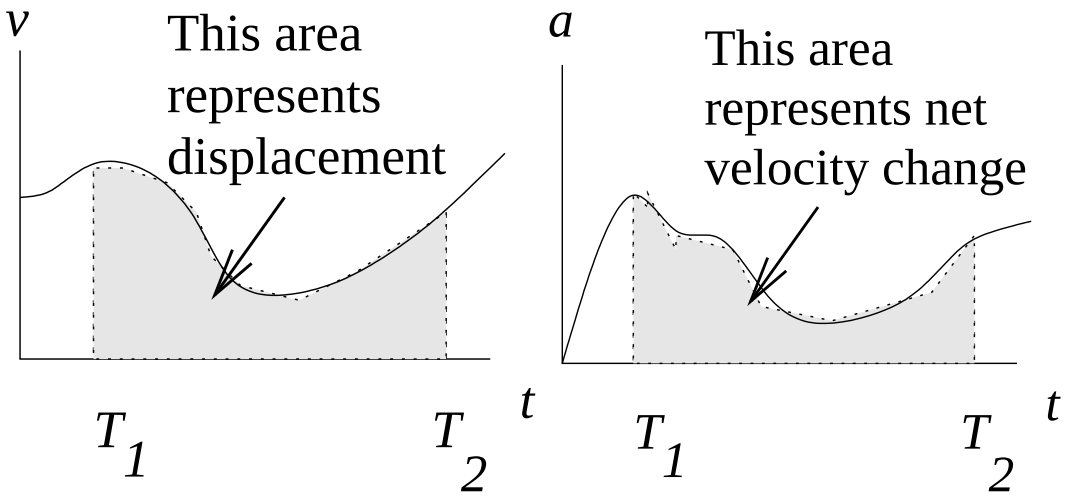


Figure 4.1: The total area under the velocity graph represents net displacement, and the total area under the graph of acceleration represents the net change in velocity over the interval $T_1 \leq t \leq T_2$.

Similarly, by previous remarks, the area under the curve $a(t)$ is a geometric quantity that represents the net change in the velocity, as shown on the right of Figure 4.1.

Displacement for uniform motion

We first examine the simplest case that the velocity is constant, i.e. $v(t) = v =$ constant. Then clearly, the acceleration is zero since $a = dv/dt = 0$ when v is constant. Thus, by direct antidifferentiation,

$$\int_0^T v\ dt = vt\Big|_0^T = v(T-0) = vT.$$

However, applying result (4.1) over the time interval $0 \le t \le T$ also leads to

$$\int_0^T v\ dt = x(T) - x(0).$$

Therefore, it must be true that the two expressions obtained above must be equal, i.e.

$$x(T) - x(0) = vT.$$

Thus, here, the displacement is proportional to the velocity and to the time elapsed, and the final position is

$$x(T) = x(0) + vT.$$

This is true for all time T, so we can rewrite the results in terms of the more familiar (lower case) notation for time, t, i.e.

$$x(t) = x(0) + vt.$$

Uniformly accelerated motion

In this case the acceleration is constant. Thus, by direct antidifferentiation,

$$\int_0^T a\ dt = at\Big|_0^T = a(T-0) = aT,$$

However, using result (4.2) for $0 \le t \le T$ leads to

$$\int_0^T a\ dt = v(T) - v(0).$$

Since these two, results must match, $v(T) - v(0) = aT$ so that

$$v(T) = v(0) + aT.$$

Let us refer to the initial velocity $V(0)$ as v_0. The above connection between velocity and acceleration holds for any final time T, i.e., it is true for all t that:

$$v(t) = v_0 + at.$$

This just means that velocity at time t is the initial velocity incremented by an increase (over the given time interval) due to the acceleration. From this we can find the displacement and position of the particle as follows: Let us call the initial position $x(0) = x_0$. Then

$$\int_0^T v(t)\ dt = x(T) - x_0.$$

But

$$I = \int_0^T v(t)\ dt = \int_0^T (v_0 + at)\ dt = \left(v_0 t + a\frac{t^2}{2}\right)\Big|_0^T = \left(v_0 T + a\frac{T^2}{2}\right).$$

So, setting these two results equal means that

$$x(T) - x_0 = v_0 T + a\frac{T^2}{2}.$$

But this is true for *all* final times, T, i.e. this holds for any time t so that

$$x(t) = x_0 + v_0 t + a\frac{t^2}{2}.$$

This expression represents the position of a particle at time t given that it experienced a constant acceleration, a, after starting with initial velocity v_0, from initial position x_0 at time $t = 0$.

Non-constant acceleration and terminal velocity

The velocity of a sky-diver is described by a **differential equation**

$$\frac{dv}{dt} = g - kv$$

i.e., a mathematical statement that relates changes in velocity to acceleration due to gravity, g, and drag forces due to friction with the atmosphere. A good approximation for such drag forces is the term kv, proportional to the velocity, with k, a positive constant, representing a frictional coefficient. We will assume that initially the velocity is zero, i.e. $v(0) = 0$. In this example we show how to determine the velocity at any time t. In this calculation, we will use the following result from first term calculus material:

The differential equation and initial condition

$$\frac{dy}{dt} = -ky, \quad y(0) = 0 \tag{4.3}$$

has a solution

$$y(t) = y_0 e^{-kt} \tag{4.4}$$

The equation $dv/dt = g - kv$ implies that

$$a(t) = g - kv(t)$$

where $a(t)$ is the acceleration at time t. Taking a derivative of both sides of this equation leads to

$$\frac{da}{dt} = -k\frac{dv}{dt},$$

which means that

$$\frac{da}{dt} = -ka.$$

We observe that this equation is precisely like equation (4.3) (with a replacing y), which implies (according to 4.3) that $a(t)$ is given by

$$a(t) = C\ e^{-kt} = a_0\ e^{-kt}.$$

Initially, at time $t = 0$, the acceleration is $a(0) = g$ (since $a(t) = g - kv(t)$, and we have assumed that $v(0) = 0$). Then

$$a(t) = g\ e^{-kt}.$$

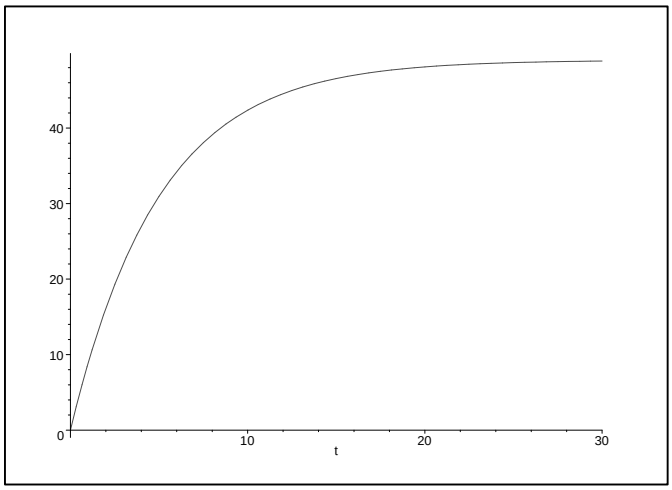


Figure 4.2: Terminal velocity (m/s) for acceleration due to gravity g=9.8 m/s^2, and $k = 0.2$. The velocity reaches a near constant 49 m/s after about 20 s.

Then

$$\int_0^T a(t)\ dt = \int_0^T g\ e^{-kt}\ dt = g\ \int_0^T e^{-kt}\ dt = g\ \left[\frac{e^{-kt}}{-k}\right]\Bigg|_0^T = g\ \frac{(e^{-kT} - 1)}{-k} = \frac{g}{k}\ \left(1 - e^{-kT}\right).$$

As before, based on equation (4.2) the same integral of acceleration over $0 \le t \le T$ must equal $v(T) - v(0)$. But $v(0) = 0$ by assumption, and the result is true for *any* final time T, so, in particular, setting $T = t$, and combining both results leads to an expression for the velocity at any time:

$$v(t) = \frac{g}{k}\ \left(1 - e^{-kt}\right).$$

We can also determine how the velocity behaves in the long term: observe that for $t \to \infty$, the exponential term $e^{-kt} \to 0$, so that

$$v(t) \to \frac{g}{k}(1 - \text{ very small quantity}) \approx \frac{g}{k}$$

Thus, when drag forces are in effect, the falling object does not continue to accelerate indefinitely: it eventually attains a *terminal velocity*. We have seen that this limiting velocity is $v = g/k$. The object continues to fall at this (approximately constant) speed as shown in Figure 4.2.

4.2 From density to total number

The concept of *density* is used widely in many applications in science. In this section, we give several examples of the connection between density and total amount. As discussed below, that connection is closely related to the idea of integration.

4.2.1 Density of cars along a highway

A measure of highway congestion might be the *density* of cars along a stretch of read. By density we here mean *number of cars per unit distance*. Suppose we want to make a connection between the total number of vehicles on a particular segment of the highway and their density along that segment. If the cars are spaced equal distances apart, this would be easy: Total number = density per unit distance times the total length of the segment of road. However, if the cars are distributed non-uniformly, we need a better approach. Suppose we let x denote position along the highway, with $x = 0$ at the beginning of the segment of interest. Suppose $C(x)$ is the density of cars at position x.

We might subdivide the road up into small sections each of length Δx, and approximate the number of cars on each of these small segments by $C(x)\Delta x$. If those segments are small enough, this should be a relatively good approximation. The *Total* number of cars *all along* the road would then be

$$\sum C(x)\Delta x$$

where the sum extends from $x = 0$ to the end of the highway, at some place $x = L$. From this idea, we find that there is a connection between the total number and the integral of the density, i.e., that

$$\int_0^L C(x)\ dx = \text{Total} \quad \text{number} \quad \text{of} \quad \text{cars.}$$

Example

At rush hour, the density of cars along a highway is given by

$$C(x) = 100x(1 - \frac{x}{10}) \quad 0 \leq x \leq L$$

where x is distance in kilometers

(a) What is the largest value of L for which this density makes sense?

(b) Where along the highway is the congestion greatest? What is the car density at that location?

(c) What is the total number of cars along the road?

Solution

(a) Density has to be a positive quantity. Therefore $L = 10$ is the largest allowable value of x. This formula would break down outside of the range $0 \leq x \leq 10$.

(b) Greatest highest congestion is equivalent to greatest density. To find this, we look for a maximum in the function $C(x)$, e.g. by setting its derivative to zero. We find that $C'(x) = 100 - 20x = 0$ leads to $x = 5$, so the greatest congestion is in the middle of the road, at $x = 5$ km. At this location, we have $C(5) = 250$ cars per kilometer.

(c) The total number of cars on the road (assuming that we take $L = 10$) is

$$I = \int_0^L C(x)\ dx = \int_0^L 100x(1 - \frac{x}{10})\ dx = \int_0^{10} (100x - 10x^2)\ dx.$$

We calculate the above integral:

$$I = 100\left(\frac{x^2}{2} - \frac{1}{10}\frac{x^3}{3}\right)\Big|_0^{10} = 10^4\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{10^4}{6} = 1,666$$

Thus, there are 1,666 cars over the ten kilometers highway at rush hour.

4.3 From rates of change to total change

In this section, we examine several examples in which the rate of change of some process is specified. We use this information to obtain the net (total) change that occurs over some time period.

Changing temperature

We must carefully distinguish between information about the time dependence of some function, from information about the rate of change of some function. Here is an example of these two different cases, and how we would handle them

(a) The temperature of a cup of juice is observed to be

$$T(t) = 25(1 - e^{-0.1t})$$

in degrees Celsius where t is time in minutes. Find the change in the temperature of the juice between $t = 1$ and $t = 5$.

Solution

(a) In this case, we are given the temperature as a function of time. To determine what **net change** occurred between times $t = 1$ and $t = 5$, we would find the temperatures at each time point and subtract: That is, the change in temperature between times $t = 1$ and $t = 5$ is simply

$$T(5) - T(1) = 25(1 - e^{-0.5}) - 25(1 - e^{-0.1}) = 25(0.94 - 0.606) = 7.47.$$

(b) The **rate of change** of temperature of a cup of coffee is observed to be

$$f(t) = 8e^{-0.2t}$$

degree Celsius per minute where t is time in minutes. What is the **total change** in the temperature between $t = 1$ and $t = 5$ minutes ?

Solution

(b) Here, we do not know the temperature at any time, but we are given information about **the rate of change**. (Carefully note the subtle difference in the wording.) The total change is found by integrating the rate of change, $f(t)$, from $t = 1$ to $t = 5$, i.e. by calculating

$$I = \int_1^5 f(t)\ dt = \int_1^5 8e^{-0.2t}dt = -40e^{-0.2t}\Big|_1^5 = -40e^{-1} + 40e^{-0.2}$$

$$I = 40(e^{-0.2} - e^{-1}) = 40(0.8187 - 0.3678) = 18.$$

4.3.1 Tree growth rates

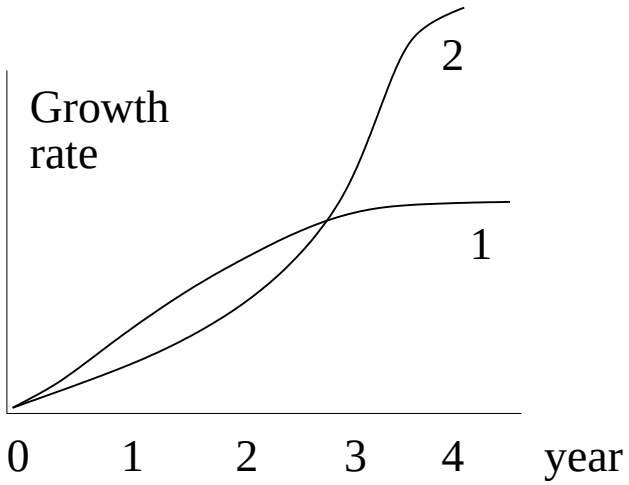


Figure 4.3: Growth rates of two trees.

The rate of growth in height for two species of trees (in feet per year) is shown in Figure 4.3. If the trees start at the same height, which tree is taller after 1 year? After 4 years?

Solution

We recognize that the net change in height of each tree is of the form

$$H_i(T) - H_i(0) = \int_0^T g_i(t)dt,$$

where $g_i(t)$ is the growth rate as a function of time (shown for each tree in Figure 4.3) and $H_i(t)$ is the height of tree "i" at time t. But, by the Fundamental Theorem of Calculus, this definite integral corresponds to the area under the curve $g_i(t)$ from $t = 0$ to $t = T$. Thus we must interpret the net change in height for each tree as the area under its growth curve. We see from Figure 4.3 that at 1 year, the area under the curve for tree 1 is greater, so it has grown more. After 4 years the area under the second curve is greatest so tree 2 has grown most by that time.

4.3.2 Radius of a tree trunk

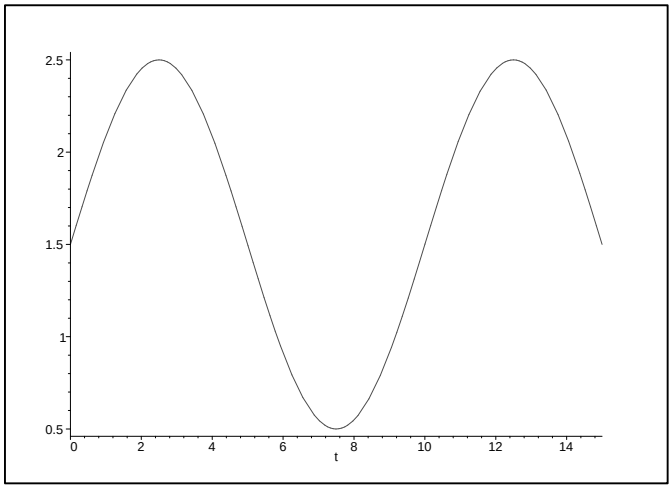


Figure 4.4: Rate of change of radius, $f(t)$ for a growing tree.

The trunk of a tree, assumed to have the shape of a cylinder, grows incrementally, so that its cross-section consists of "rings". In years of plentiful rain and adequate nutrients, the tree grows faster than in years of drought or poor soil conditions. Suppose the rainfall pattern has been cyclic, so that, over a period of 10 years, the growth rate of the radius of the tree trunk (in cm/year) is given by the function

$$f(t) = 1.5 + \sin(\pi t/5),$$

shown in Figure 4.4. Let the height of the tree (and its trunk) be approximately constant over this ten year period, and assume that the density of the trunk is approximately 1 gm/cm^3.

(a) If the radius was initially r_0 at time $t = 0$, what will the radius of the trunk be at time t later?

(b) What will be the ratio of its final and initial mass?

Solution

(a) The rate of change of the radius of the tree is given by the function $f(t)$, and we are told that at $t = 0$, $R(0) = r_0$. A graph of this growth rate over the first fifteen years is shown in Figure 4.4. The net change in the radius is

$$R(t) - R(0) = \int_0^t f(s)\, ds = \int_0^t (1.5 + \sin(\pi s/5))\, ds.$$

Integrating the above, we get

$$R(t) - R(0) = \left(1.5t - \frac{\cos(\pi s/5)}{\pi/5}\right)\Bigg|_0^t.$$

Thus, using the initial value, $R(0) = r_0$ (which is a constant), and evaluating the integral, leads to

$$R(t) = r_0 + 1.5t - \frac{5\cos(\pi t/5)}{\pi} + \frac{5}{\pi}.$$

(The constant at the end of the expression stems from the fact that $\cos(0) = 1$.) A graph of the radius of the tree over time (using $r_0 = 1$) is shown in Figure 4.5. This function is equivalent to the area associated withe the function shown in Figure 4.4. Notice that Figure 4.5 confirms that the radius keeps growing over the entire period, but that its growth rate (slope of the curve) alternates between higher and lower values.

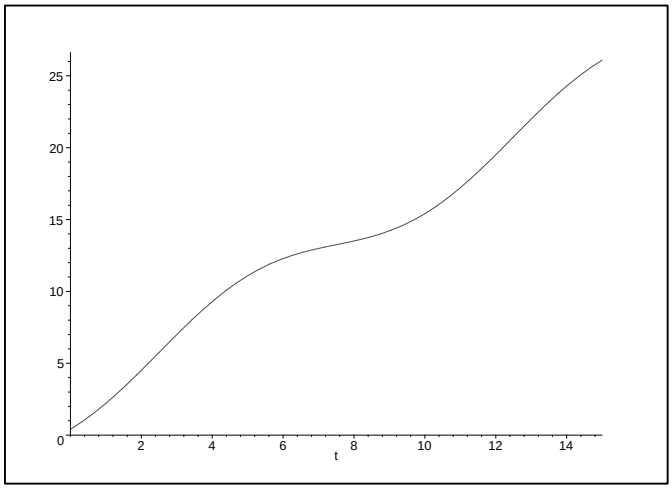


Figure 4.5: Rate of change of radius, $f(t)$ for a growing tree.

After ten years we have

$$R(10) = r_0 + 15 - \frac{5}{\pi}\cos(2\pi) + \frac{5}{\pi}.$$

But $\cos(2\pi) = 1$, so

$$R(10) = r_0 + 15.$$

(b) The mass of the tree is density times volume, and since the density is taken to be 1 gm/cm^3, we need only obtain the initial and final volume. Taking the trunk to be cylindrical means that the volume at a given time is

$$V(t) = \pi R(t)^2 h.$$

The ratio of final and initial volumes (and hence the ratio of final and initial mass) is

$$\frac{V(10)}{V(0)} = \frac{\pi[R(10)]^2 h}{\pi r_0^2 h} = \frac{[R(10)]^2}{r_0^2} = \left(\frac{r_0 + 15}{r_0}\right)^2.$$

4.3.3 Birth rates and total births

After World War II, the birth rate in western countries increased dramatically. Suppose that the number of babies born (in millions per year) was given by

$$b(t) = 5 + 2t, \quad 0 \leq t \leq 10$$

where t is time in years after the end of the war.

(a) How many babies in total were born during this time period (i.e in the first 10 years after the war)?

(b) Find the time T_0 such that the total number of babies born from the end of the war up to the time T_0 was precisely 14 million.

Solution

(a) To find the number of births, we would integrate the birth rate, $b(t)$ over the given time period. The *net change* in the population due to births (neglecting deaths) is

$$P(10) - P(0) = \int_0^{10} b(t)\ dt = \int_0^{10} (5 + 2t)\ dt = (5t + t^2)|_0^{10} = 50 + 100 = 150[\text{million babies}].$$

(b) Denote by T the time at which the total number of babies born was 14 million. Then, [in units of million]

$$I = \int_0^T b(t)\ dt = 14$$

$$I = \int_0^T (5 + 2t)\ dt = 5T + T^2$$

But $\Rightarrow 5T + T^2 = 14 \Rightarrow T^2 + 5T - 14 = 0 \Rightarrow (T - 2)(T + 7) = 0$. This has two solutions, but we reject $T = -7$ since we are looking for time after the War. Thus we find that $T = 2$ years, i.e it took two years for 14 million babies to have been born.

4.4 Production and removal

The process of integration can be used to convert rates of production and removal into net amounts present at a given time. The example in this section is of this type. We investigate a process in which a substance accumulates as it is being produced, but disappears through some removal process. We would like to determine when the quantity of material increases, and when it decreases. Here we will combine both quantitative calculations and qualitative graph-interpretation skills.

Circadean rhythm in hormone levels

Consider a hormone whose level in the blood at time t will be denoted by $H(t)$. We will assume that the level of hormone is regulated by two separate processes: one might be the secretion rate of specialized cells that produce the hormone. (The production rate of hormone might depend on the time of day, in some cyclic pattern that repeats every 24 hours or so.) This type of cyclic pattern is called *circadean* rhythm. A competing process might be the removal of hormone (or its active deactivation by some enzymes secreted by other cells.) In this example we will assume that both the production rate, $p(t)$, and the removal rate, $r(t)$ of the hormone are time-dependent, periodic functions with somewhat different phases.

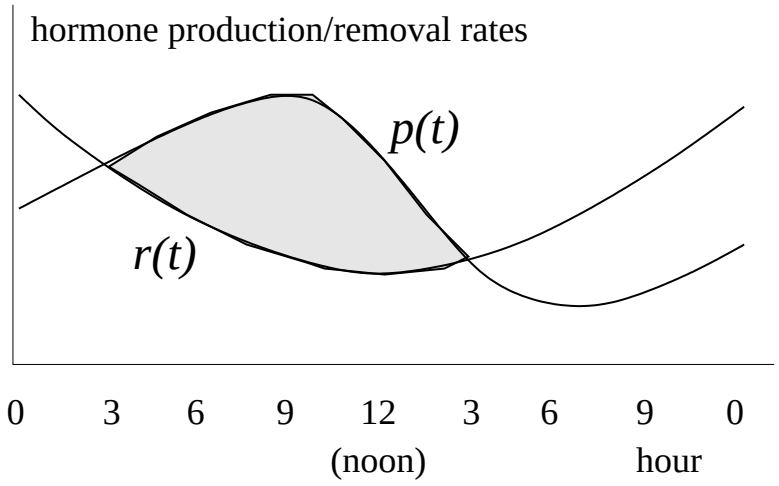


Figure 4.6: The rate of hormone production $p(t)$ and the rate of removel $r(t)$ are shown here. We want to use these graphs to deduce when the level of hormone is highest and lowest.

A typical example of two such functions are shown in Figure 4.6. This figure shows the production and removal rates over a period of 24 hours, starting at midnight. Our first task will be to use properties of the graph in Figure 4.6 to answer the following questions:

1. When is the production rate, $p(t)$, maximal?
2. When is the removal rate $r(t)$ minimal?
3. At what time is the hormone level in the blood highest?
4. At what time is it lowest?
5. Find the maximal level of hormone in the blood over the period shown, assuming that its basal (lowest) level is $H = 0$.

Solutions

1. Production rate is maximal at about 9:00 am.

2. Removal rate is minimal at noon.

3. To answer this question we note that the total amount of hormone produced over a time period $a \leq t \leq b$ is

$$P_{\text{total}} = \int_a^b p(t)dt.$$

The total amount removed over time interval $a < t < b$ is

$$R_{\text{total}} = \int_a^b r(t)dt.$$

This means that the net change in hormone level over the given time interval (amount produced minus amount removed) is

$$H(b) - H(a) = P_{\text{total}} - R_{\text{total}} = \int_a^b (p(t) - r(t))dt.$$

We interpret this integral as the *area between the curves* $p(t)$ and $r(t)$. But we must use caution here: For any time interval over which $p(t) > r(t)$, this integral will be positive, and the hormone level will have increased. Otherwise, if $r(t) < p(t)$, the integral yields a negative result, so that the hormone level has decreased. This makes simple intuitive sense: If production is greater than removal, the level of the substance is accumulating, whereas in the opposite situation, the substance is decreasing. With these remarks, we find that the hormone level in the blood will be *greatest* at 3:00 pm, when the greatest (positive) area between the two curves has been obtained.

4. Similarly, the least hormone level occurs after a period in which the removal rate has been larger than production for the longest stretch. This occurs at 3:00 am, just as the curves cross each other.

5. Here we will practice integration by actually fitting some cyclic functions to the graphs shown in the Figure. Our first observation is that if the length of the cycle (also called the *period*) is 24 hours, then the *frequency* of the oscillation is $\omega = (2\pi)/24 = \pi/12$. We further observe that the functions shown in the Figure 4.7 have the form

$$p(t) = A(1 + \sin(\omega t)), \quad r(t) = A(1 + \cos(\omega t)).$$

Intersection points occur when

$$p(t) = r(t)$$

$$A(1 + \sin(\omega t)) = A(1 + \cos(\omega t)),$$

$$\sin(\omega t) = \cos(\omega t)),$$

$$\Rightarrow \tan(\omega t) = 1.$$

This last equality leads to $\omega t = \pi/4, 5\pi/4$. But then, using the fact that $\omega = \pi/12$ we find that $t = 3, 15$. Thus, over the time period $3 \leq t \leq 15$ hrs, the hormone level is increasing. We can now compute the net increase in hormone over this period. For simplicity, we will take

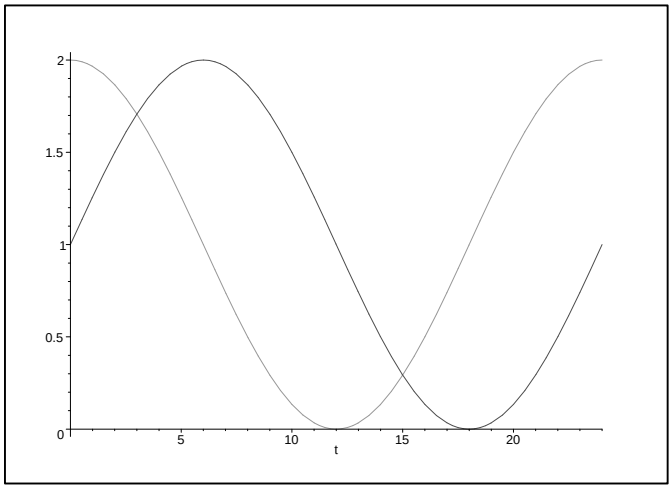


Figure 4.7: The functions shown above are trigonometric approximations to the rates of hormone production and removal from Figure 4.6

the amplitude $A = 1$ from here on. In general, this would just be a multiplicative constant in whatever solution we compute We find that

$$H_{\text{total}} = \int_3^{15} [p(t) - r(t)]\, dt = \int_3^{15} [(1 + \sin(\omega t)) - (1 + \cos(\omega t))]\, dt$$

$$H_{\text{total}} = \int_3^{15} \left(\sin \frac{\pi}{12} t - \cos \frac{\pi}{12} t\right) dt = \left(-\frac{\cos \frac{\pi}{12} t}{\pi/12} - \frac{\sin \frac{\pi}{12} t}{\pi/12}\right) \Bigg|_3^{15}$$

$$H_{\text{total}} = -\frac{12}{\pi} \left((\cos \frac{15\pi}{12} + \sin \frac{15\pi}{12}) - (\cos \frac{3\pi}{12} + \sin \frac{3\pi}{12})\right)$$

After some simplifications, we find that

$$H_{\text{total}} = -\frac{12}{\pi} \left(-\frac{4\sqrt{2}}{2}\right) = 24\frac{\sqrt{2}}{\pi}.$$

Thus the amount of hormone that accumulated over the time interval $3 \leq t \leq 15$, i.e. between 3:00 am and 3:00 pm is $24\sqrt{2}A\pi$.

4.5 Average value of a function

Given a function

$$y = f(x)$$

over some interval $a < x < b$, we will define average value of the function as follows:

Definition

The average value of $f(x)$ over the interval $a \leq x \leq b$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)dx.$$

Remark

This type of average should *not* be confused with the mean or average of a distribution in the context of probability, to be seen later on in this course.

Example 1

Find the average value of the function

$$y = f(x) = x^2$$

over the interval $2 < x < 4$.

Solution

$$\bar{f} = \frac{1}{4-2} \int_2^4 x^2 \, dx = \frac{1}{2} \frac{x^3}{3} \Big|_2^4$$

$$\bar{f} = \frac{1}{6} \left(4^3 - 2^3\right) = \frac{28}{3}$$

Example 2: Day length over the year

Suppose we want to know the average length of the day during summer and spring. We will assume that day length follows a simple periodic behaviour, with a cycle length of 1 year (365 days). Let us measure time t in days, with $t = 0$ at the spring equinox, i.e. the date in spring when night and day lengths are equal. (On that data, the length of the day is 12 hrs.) We will refer to the number of daylight hours on day t by the function $f(t)$. (We will also call $f(t)$ the length (in hours) of the day during a given part of the year.)

A simple function that would describe the cyclic changes of day length over the seasons is

$$f(t) = 12 + 4\sin(\frac{2\pi t}{365})$$

The average day length over spring and summer, i.e. over the first $(365/2) \approx 182$ days is:

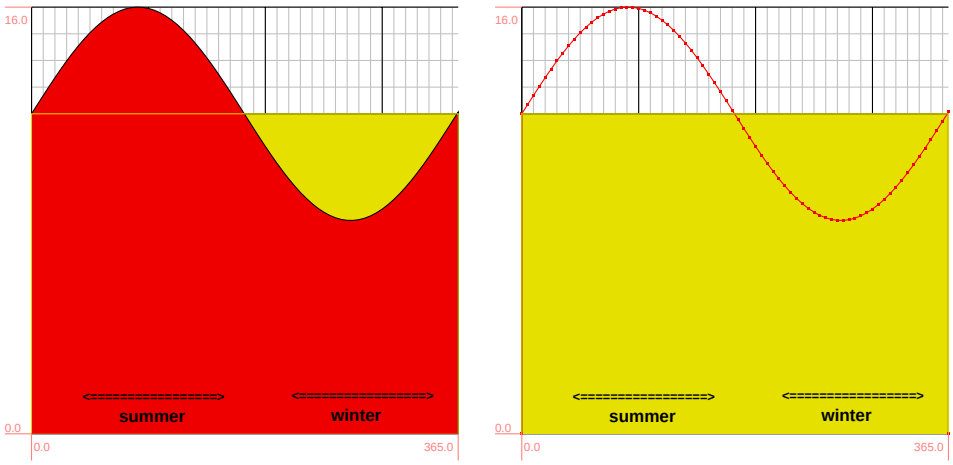


Figure 4.8: We show the variations in day length (cyclic curve) as well as the average day length (height of rectangle) in this figure.

$$\begin{aligned}
\bar{f} &= \frac{1}{182}\int_0^{182} f(t)dt && (4.5)\\
&= \frac{1}{182}\int_0^{182}\left(12+4\sin(\frac{\pi t}{182})\right)\,dt && (4.6)\\
&= \frac{1}{182}\left(12t-\frac{4\cdot 182}{\pi}\cos(\frac{\pi t}{182})\right)\Big|_0^{182} && (4.7)\\
&= \frac{1}{182}\left(12\cdot 182-\frac{4\cdot 182}{\pi}[\cos(\pi)-\cos(0)]\right) && (4.8)\\
&= 12+\frac{8}{\pi}=14.546 && (4.9)
\end{aligned}$$

Thus, on average, the day is 14.546 hrs long during the spring and summer.

In Figure 4.8, we show the entire day length cycle over one year. It is left as an exercise for the reader to show with a similar calculation that the average value of f over the entire year is 12 hrs. (This makes intuitive sense, since overall, the short days in winter will average out with the longer days in summer.)

Figure 4.8 also shows geometrically what the average value of the function represents. Suppose we determine the area associated with the graph of $f(x)$ over the interval of interest. (This area is painted red (dark) in Figure 4.8, where the interval is $0 \le t \le 365$, i.e. the whole year.) Now let us draw in a rectangle over the same interval ($0 \le t \le 365$) having the same total area. (See big rectangle in Figure 4.8, and note that its area matches with the darker, red region.) The height of the rectangle represents the average value of $f(t)$ over the interval of interest.