## Degenerate Pivots and Cycling

A pivot in the Simplex Method is said to be **degenerate** when it doesn't change the basic solution. This happens when we get a ratio of 0 in choosing the leaving variable.

Degenerate pivots are quite common, and usually harmless. But it's possible for **cycling** to occur in a sequence of degenerate pivots. This means that the same tableau occurs more than once. When that happens, the Simplex Method would keep repeating a sequence of pivots forever.

The following example exhibits cycling, using our standard pivoting rules.

$$\begin{array}{ll} \text{maximize} & z = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} & 1/2x_1 - 11/2x_2 - 5/2x_3 + 9x_4 \le 0 \\ & 1/2x_1 - 3/2x_2 - 1/2x_3 + x_4 \le 0 \\ & x_1 + x_2 + x_3 + x_4 \le 1 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

Initial tableau:

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −10 | 57 | 9 | 24 | 0 | 0 | 0 | 0 | = | $z$ |
| 0 | 1/2 | −11/2 | −5/2 | 9 | 1 | 0 | 0 | 0 | = | $s_1$ |
| 0 | 1/2 | −3/2 | −1/2 | 1 | 0 | 1 | 0 | 0 | = | $s_2$ |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | = | $s_3$ |

$x_1$ enters, $s_1$ leaves (in a tie for smallest ratio).

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | −53 | −41 | 204 | 20 | 0 | 0 | 0 | = | $z$ |
| 0 | 1 | −11 | −5 | 18 | 2 | 0 | 0 | 0 | = | $x_1$ |
| 0 | 0 | 4 | 2 | −8 | −1 | 1 | 0 | 0 | = | $s_2$ |
| 0 | 0 | 12 | 6 | −17 | −2 | 0 | 1 | 1 | = | $s_3$ |

$x_2$ enters (since $-53 < -41$), $s_2$ leaves

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | −29/2 | 98 | 27/4 | 53/4 | 0 | 0 | = | $z$ |
| 0 | 1 | 0 | 1/2 | −4 | −3/4 | 11/4 | 0 | 0 | = | $x_1$ |
| 0 | 0 | 1 | 1/2 | −2 | −1/4 | 1/4 | 0 | 0 | = | $x_2$ |
| 0 | 0 | 0 | 0 | 7 | 1 | −3 | 1 | 1 | = | $s_3$ |

$x_3$ enters, $x_1$ leaves (another tie for smallest ratio)

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 29 | 0 | 0 | −18 | −15 | 93 | 0 | 0 | = | $z$ |
| 0 | 2 | 0 | 1 | −8 | −3/2 | 11/2 | 0 | 0 | = | $x_3$ |
| 0 | −1 | 1 | 0 | 2 | 1/2 | −5/2 | 0 | 0 | = | $x_2$ |
| 0 | 0 | 0 | 0 | 7 | 1 | −3 | 1 | 1 | = | $s_3$ |

$x_4$ enters ($-18 < -15$), $x_2$ leaves

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 20 | 9 | 0 | 0 | −21/2 | 141/2 | 0 | 0 | = | $z$ |
| 0 | −2 | 4 | 1 | 0 | 1/2 | −9/2 | 0 | 0 | = | $x_3$ |
| 0 | −1/2 | 1/2 | 0 | 1 | 1/4 | −5/4 | 0 | 0 | = | $x_4$ |
| 0 | 7/2 | −7/2 | 0 | 0 | −3/4 | 23/4 | 1 | 1 | = | $s_3$ |

$s_1$ enters, $x_3$ leaves (another tie for smallest ratio)

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −22 | 93 | 21 | 0 | 0 | −24 | 0 | 0 | = | $z$ |
| 0 | −4 | 8 | 2 | 0 | 1 | −9 | 0 | 0 | = | $s_1$ |
| 0 | 1/2 | −3/2 | −1/2 | 1 | 0 | 1 | 0 | 0 | = | $x_4$ |
| 0 | 1/2 | 5/2 | 3/2 | 0 | 0 | −1 | 1 | 1 | = | $s_3$ |

$s_2$ enters, $x_4$ leaves. This takes us back to the original tableau.

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −10 | 57 | 9 | 24 | 0 | 0 | 0 | 0 | = | $z$ |
| 0 | 1/2 | −11/2 | −5/2 | 9 | 1 | 0 | 0 | 0 | = | $s_1$ |
| 0 | 1/2 | −3/2 | −1/2 | 1 | 0 | 1 | 0 | 0 | = | $s_2$ |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | = | $s_3$ |

One method of preventing cycling is to use Bland's Rule instead of the Most Negative Coefficient Rule. Given a fixed ordering of the variables, e.g. $x_1, x_2, \ldots, x_n, s_1, \ldots, s_m$, Bland's Rule says:

(a) When choosing the entering variable, take the first one in the ordering that has a negative entry in the objective row.

(b) When choosing the leaving variable, if there is a tie for least ratio, take the candidate that is first in the ordering.

If we were using Bland's Rule, everything would have been the same up to that last pivot, where $x_1$ would have entered instead of $s_2$, and $x_4$ would have left.

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 27 | −1 | 44 | 0 | 20 | 0 | 0 | = | $z$ |
| 0 | 0 | −4 | −2 | 8 | 1 | −1 | 0 | 0 | = | $s_1$ |
| 0 | 1 | −3 | −1 | 2 | 0 | 2 | 0 | 0 | = | $x_1$ |
| 0 | 0 | 4 | 2 | −1 | 0 | −2 | 1 | 1 | = | $s_3$ |

Then $x_3$ would have entered, and $s_3$ left. This would be, at last, a nondegenerate pivot, producing an optimal tableau:

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | rhs | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 29 | 0 | 87/2 | 0 | 19 | 1/2 | 1/2 | = | $z$ |
| 0 | 0 | 0 | 0 | 7 | 1 | −3 | 1 | 1 | = | $s_1$ |
| 0 | 1 | −1 | 0 | 3/2 | 0 | 1 | 1/2 | 1/2 | = | $x_1$ |
| 0 | 0 | 2 | 1 | −1/2 | 0 | −1 | 1/2 | 1/2 | = | $x_3$ |