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Markov Processes and **Related Fields**

Some Generic Results in Mathematical Physics

R.B. Israel

Department of Mathematics, University of British Columbia

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Abstract. We discuss three consequences of the Baire Category Theorem. In spaces of long-range interactions, typical phase diagrams are trivial. Most ergodic states of a lattice system are not Gibbs states. Unitarily-equivalent self-adjoint operators on Hilbert space with purely discrete spectrum usually can't be added.

KEYWORDS: generic sets, very sparse sets, generic nonGibbsianness, generic lack of self-adjointness properties

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A recurring theme in my work in analysis and mathematical physics has been identifying certain sets as "generic" or "residual" in the sense of the Baire Category Theorem, i.e. containing a dense G_δ in some space which is either a complete metric space or a compact Hausdorff space. Such results can be thought of as telling us what is "typical" in this space. It's a rather different notion of "typical" than what measure theory would give us in a case where the space has a natural measure, where we can talk about properties that are true almost everywhere. One might say it tends to be a rather pessimistic viewpoint: generically, the members of the space are as poorly behaved as possible. For example, the decimal digits of almost all real numbers have nice statistical properties: the proportion of 1's in the first n digits converges to 1/10 as $n \to \infty$ with probability 1. But for a "generic" real number the proportion won't converge: the lim sup is 1 and the lim inf is 0.

My involvement with this area began with the first problem I solved for my thesis, [2]. Barry Simon, my advisor, asked me to prove a version of the Gibbs Phase Rule for convex functions on Banach spaces. In the framework of statistical mechanics, the interactions form a Banach space $\mathcal{B}$, the pressure is a continuous convex function P on $\mathcal{B}$, and the set of equilibrium states for an interaction Φ corresponds to the subdifferential $\partial P(\Phi)$ of P at Φ. The number of pure phases for the interaction is $1+\dim(\partial P(\Phi))$. What we expect from thermodynamics is that in an n-parameter family of interactions, for each $k \leq n+1$ those with k or more pure phases form an $n+1-k$-dimensional subset, and more than $n+1$ pure phases won't occur. I proved that this behaviour is "generic" if the dimensions of these subsets is understood in the sense of Hausdorff dimension. That is, we say an n-dimensional subset G of $\mathcal{B}$ satisfies the Phase Rule if the set of $\Phi \in G$ with $\dim(\partial P(\Phi) \geq m$ has Hausdorff dimension at most $n-m$ for $1 \leq m \leq n$ and is empty for $m > n$. Given an affine subspace N_0 of $\mathcal{B}$ with dimension $< n$, let G_n be the space of n-dimensional affine subspaces containing N_0. This is a complete metric space in a natural way. Then my result was that for any continuous convex function on a separable Banach space, the subspaces satisfying the Phase Rule form a dense G_δ in G_n.

A few years later I had my first encounter with the name van Enter. Daniels and van Enter [1] showed that phase transitions were less stable than we had thought: a small but long-range perturbation of the interaction can ruin a first-order phase transition, so that you have only one equilibrium state. I was able to generalize this and produce a "generic" version of it, showing [3, 4] that in a Banach space of long-range interactions my Gibbs Phase Rule will hold in a very strong (and un-physical) way: the subspaces where all interactions have a unique equilibrium state form a residual set. In fact, more is true: take any nonempty σ-compact subset K of $\mathcal{B}$ and consider n-tuples $(\Psi_1, \ldots, \Psi_n) \in \mathcal{B}^n$ such that for every $\Phi \in K$ and every $\mathbf{c} \in \mathbf{R}^n \setminus \{0\}$, the interaction $\Phi + \sum c_i \Psi_i$ has a unique equilibrium state. Then the set of such n-tuples is residual in $\mathcal{B}^n$.

This, by the way, is a kind of strengthening of the Baire Category framework that I think might be useful in other contexts. Let's call a set S in a Banach space $\mathcal{X}$ "sparse" if for every compact subset K of $\mathcal{X}$, $S+K$ is a set of first category in $\mathcal{X}$, and "very sparse" if $\mathbf{R}S = \{ts : t \text{ real}, s \in S\}$ is sparse. The set of interactions in $\mathcal{B}$ with more than one equilibrium state is very sparse. Another example: in the space $C[0,1]$ of continuous functions on $[0,1]$, the set of those that are differentiable at some point is very sparse. On the other hand, a proper subspace of finite codimension in a Banach space is first category but not sparse. The main result on very sparse sets is this:

If S is a very sparse F_σ in a Banach space $\mathcal{X}$, then for any σ-compact subset K of $\mathcal{X}$ and positive integer n,

$$\{\mathbf{x} = (x_1, ..., x_n) \in \mathcal{X}^k : ((\mathbf{R}^n \setminus \{0\}).\mathbf{x} + K) \cap S = \emptyset\} \text{ is residual.}$$

Thus, in particular, a very sparse F_σ will not meet a "generic" finite-dimensional linear subspace (except perhaps at 0).

Now for something more closely related to the theme of this meeting: Gibbsian states. Consider a classical system on the lattice $\mathbf{Z}^d$ with configuration space Ω. For simplicity, I'll assume the single-site space Ω_0 is finite, although this assumption can be removed. Consider the set K_Ω of translation-invariant probability measures on Ω. Using the weak-$*$ topology on K_Ω as a set of linear functionals on $C(\Omega)$, K_Ω is a metrizable compact convex set. Moreover it is a Choquet simplex. The extreme points of K_Ω are the ergodic states. It turns out [2, Lemma IV.3.2] that the set $\operatorname{ex} K_\Omega$ of ergodic states is a dense G_δ in K_Ω. A member of K_Ω is Gibbsian if it is a Gibbs state for some (absolutely summable interaction). Now van Enter, Fernández and Sokal [5, Conjecture 4.19], conjectured, and I proved, that the Gibbsian states form a set of first category in K_Ω (and of course in $\operatorname{ex} K_\Omega$, since that's a dense G_δ in K_Ω). This has not been published before, and at Aernout's suggestion I'll do it here.

The basic property of Gibbsian states that we use is that they are quasilocal. In fact, what we'll show is that a certain set of non-quasilocal probability measures is a dense G_δ. For a quasilocal probability measure ρ, the conditional probabilities

$$\rho_\Lambda(x|y) = \rho(\omega_\Lambda = x \,|\, \omega_{\Lambda^c} = y)$$

are continuous functions of $y \in \Omega_{\Lambda^c} = (\Omega_0)^{\Lambda^c}$ for each finite subset Λ of the lattice and each $x \in \Omega_\Lambda = (\Omega_0)^\Lambda$. In particular, since functions depending on only finitely many sites are uniformly dense in $C(\Omega_{\Lambda^c})$, for each $\varepsilon > 0$, finite set Λ and $x \in \Omega_\Lambda$ there must be a finite set Λ' containing Λ such that $\rho_\Lambda(x|y)$ is determined up to ε by the restriction of y to Λ'. A quasilocal state could not have the following property (Q):

> For every finite set Λ' containing 0, and every $\delta > 0$, there is some site $p \notin \Lambda'$ such that $\rho_0(\sigma_0 = \sigma_p | \sigma_{\Lambda' \setminus \{0\}} = y) > 1 - \delta$ for all $y \in \Omega_{\Lambda' \setminus \{0\}}$.

Thus for a state having property (Q), no matter what is happening near site 0 you can look at some site p far away and have a very good chance that the state at 0 is the same as at p.

The inequality above can be written as

$$\rho(\sigma_0 = \sigma_p, \sigma_{\Lambda' \setminus \{0\}} = y) > (1 - \delta)\rho(\sigma_{\Lambda' \setminus \{0\}} = y). \tag{1}$$

For any given δ, Λ', y and p, this is an inequality of the form $\rho(f) > \rho(g)$ where f and g are in $C(\Omega)$, and there are only finitely many $y \in \Omega_{\Lambda' \setminus \{0\}}$, so the set $U(\delta, \Lambda', p)$ of states ρ satisfying it for all y is open, and the set of states satisfying property (Q) is a G_δ. It thus suffices to show:

> For any δ and Λ', the set $U(\delta, \Lambda') = \bigcup_p U(\delta, \Lambda', p)$ of $\rho \in K(\Omega)$ such that for some $p \notin \Lambda'$, inequality (1) holds, is dense in $K(\Omega)$.

Let V be any neighbourhood of some $\rho_0 \in K(\Omega)$. We may assume ρ_0 is non-null in Λ', i.e. $\rho_0(\sigma'_\Lambda = y) > 0$ for all $y \in \Omega'_\Lambda$. For each positive integer N, let

ρ_N be a measure that agrees with ρ_0 on observables measurable in a cube C_N of side N centred at 0, but is concentrated on configurations that are periodic with period N in each coordinate direction. To get a translation-invariant measure, average ρ_N over all translations, obtaining ρ'_N. Note that for any translation τ and any observable f measurable in a finite set Λ, $\tau(\rho_N)(f) = \rho_0(f)$ if all of Λ is contained in one of the translates $\tau(C_N) + Nj$ for $j \in \mathbf{Z}^d$, and as $N \to \infty$ the fraction of translates for which this is true goes to 1. Thus $\rho'_N \to \rho_0$ in the weak-$*$ topology, and in particular $\rho'_N \in V$ if N is large enough. Also $\rho'_N(\sigma_{\Lambda'} = y) > 0$ for all $y \in \Omega'_\Lambda$ if N is sufficiently large. If $p \in N\mathbf{Z}^d \setminus \{0\}$ we have $\rho'_N(\sigma_0 = \sigma_p) = 1$ so

$$\rho'_N(\sigma_0 = \sigma_p, \sigma_{\Lambda' \setminus \{0\}} = y) = \rho'_N(\sigma_{\Lambda' \setminus \{0\}} = y) > (1 - \delta)\rho'_N(\sigma_{\Lambda' \setminus \{0\}} = y).$$

The final generic result I'd like to mention deals with operators on a Hilbert space. It arose from a remark by a physicist that for any linear operator T you can write $T = A + iB$ where $A = (T + T^*)/2$ and $B = (T - T^*)/(2i)$ are Hermitian. The operators of physics are generally unbounded (presumably closed and densely defined), and the problem with this is that there's no reason for the domains $\mathcal{D}(T)$ and $\mathcal{D}(T^*)$ to coincide, or even to have anything other than 0 in their intersection. It's not obvious how to get an example of this phenomenon, so I thought of doing a "generic" version. The key to it is the following

Theorem. *Let K_1 and K_2 be σ-compact sets in an infinite-dimensional Hilbert space $\mathcal{H}$. Then the set G of unitary operators U such that $K_1 \cap U(K_2) \subseteq \{0\}$ is a dense G_δ in the group $\mathcal{U}$ of unitary operators on $\mathcal{H}$.*

Proof. It suffices to prove that in the case where K_1 and K_2 are compact and contained in $\{x : \|x\| \geq 1\}$, G is open and dense. That it's open is easy. To show it's dense, let V be any unitary operator and $\varepsilon > 0$. I will find $U \in G$ with $\|U - V\| < \varepsilon$. Cover VK_1 and K_2 by finitely many open balls $B_\delta(x_i)$, $i = 1, \ldots, n$, of radius $\delta > 0$. The points x_i lie in a finite-dimensional space L with orthonormal basis $\{p_j : j = 1 \ldots k\}$. Take an orthonormal set $\{q_j : j = 1 \ldots k\}$ of the same cardinality in the orthogonal complement of L. Let W_t be the unitary operator taking p_j to $\cos(t)p_j + \sin(t)q_j$, q_j to $-\sin(t)p_j + \cos(t)q_j$, and leaving fixed everything orthogonal to the span of the p_j and q_j. Then $\|W_tV - V\| = \|W_t - I\| < t$, and $\|W_tx_i - x_j\| \geq \|W_tx_i - L\| \geq |\sin(t)|$. Then if we take $0 < \delta < 1/2\sin(t)$ and $t < \varepsilon$, W_tV will be the required U.

Corollary. *Let H be any self-adjoint operator on a separable infinite-dimensional Hilbert space with purely discrete spectrum. Let $T = UH$ where U is a unitary operator. Then the set of U for which $\mathcal{D}(T) \cup \mathcal{D}(T^*) = \{0\}$ is a dense G_δ.*

Since the spectral projections of H for bounded sets are finite-dimensional, $\{x \in \mathcal{D}(H) : \|x\| \leq 1 \text{ and } \|Hx\| \leq 1\}$ is compact. Therefore $\mathcal{D}(H)$ is σ-compact, and since $\mathcal{D}(T) = \mathcal{D}(H)$ while $\mathcal{D}(T^*) = U\mathcal{D}(H)$, the result follows from the theorem.

Also of possible interest is the fact that $\mathcal{D}(H) \cap \mathcal{D}(UHU^*) = \{0\}$ (so "typically" you can't add two unitarily-equivalent self-adjoint operators with the same purely discrete spectrum).

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