

# TIME OF THE WINNING GOAL

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ABSTRACT. We investigate the most likely time for the winning goal in an ice hockey game. If at least one team has a low expected score, the most likely time is at the beginning of the game; otherwise it is at some time during the game.

## 1. INTRODUCTION

A local radio station once had a contest which involved guessing in which minute of an ice hockey game the winning goal would be scored. This makes an interesting exercise in probability theory, to calculate an optimal strategy for entries in such contests. Since this is theoretical probability rather than practical statistics, we make some simplifying assumptions:

Assume each team's goals occur by independent Poisson processes with constant rates  $\lambda_1$  and  $\lambda_2$  for time  $0 \leq t \leq 1$  (representing "regulation time" - we'll ignore overtime). If the final score is  $m$  to  $n$  with  $m > n$ , then the winning goal is the  $(n + 1)$ 'th goal scored by the winning team. Let  $T$  be the time of the winning goal. We want to find the density  $f(t)$  for  $T$ , and to determine at what time  $f(t)$  is greatest.

Note that  $T$  will be left undefined if regulation time ends in a draw. Thus the integral of the density from 0 to 1 will not be 1, but rather the probability that regulation time does not end in a draw.

## 2. THE DENSITY

The best way to approach this problem seems to be to condition on the number of goals scored by one of the teams. What is the conditional probability that team 1 scores the winning goal in the time interval  $[t, t + h)$  where  $h$  is small, given that team 2 scores a total of exactly  $m$  goals? Neglecting the event that team 1 scores more than one goal in this interval (which has probability  $o(h)$ ), this happens if and only if team 1 scores exactly  $m$  goals in  $[0, t)$  and at least one in  $[t, t + h)$ . Thus the conditional probability is

$$\lambda_1 h e^{-\lambda_1 t} \frac{(\lambda_1 t)^m}{m!} + o(h)$$

Since the probability of team 2 scoring  $m$  goals in the game is  $\exp(-\lambda_2)\lambda_2^m/m!$ , we obtain the density

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$$f_1(t) = \lambda_1 e^{-\lambda_1 t - \lambda_2} \sum_{m=0}^{\infty} \frac{(\lambda_1 \lambda_2 t)^m}{(m!)^2} = \lambda_1 e^{-\lambda_1 t - \lambda_2} I_0(2\sqrt{\lambda_1 \lambda_2 t})$$

for team 1 scoring the winning goal at time  $t$ , where  $I_0$  is a modified Bessel function of order 0.

Adding the similar density for team 2, we find the total density for the winning goal being scored at time  $t$  is

$$f(t) = (\lambda_1 e^{-\lambda_1 t - \lambda_2} + \lambda_2 e^{-\lambda_1 - \lambda_2 t}) I_0(2\sqrt{\lambda_1 \lambda_2 t})$$

The integral of this for  $t$  from 0 to 1 is the probability that regulation time does not end in a draw, which is

$$1 - e^{-\lambda_1 - \lambda_2} \sum_{m=0}^{\infty} \frac{(\lambda_1 \lambda_2)^m}{(m!)^2} = 1 - e^{-\lambda_1 - \lambda_2} I_0(2\sqrt{\lambda_1 \lambda_2})$$

Now the question is to find where  $f(t)$  attains its maximum on the interval  $0 \leq t \leq 1$ . It is extremely unlikely that this can be found in general in closed form, but some qualitative results are possible. In particular we can investigate whether the maximum will occur at an interior point or an endpoint.

### 3. EVENLY MATCHED TEAMS

We first consider the special case of two evenly-matched teams:  $\lambda_1 = \lambda_2$  (which we may as well call  $\lambda$ ).

**Theorem 1.** *If  $0 < \lambda \leq 1$ , the maximum is at  $t = 0$ ; if  $\lambda > 1$  it is in  $(0, 1)$ , and is the only critical point in that interval.*

*Proof.* We have

$$f(t) = 2\lambda e^{-(1+t)\lambda} I_0(2\lambda\sqrt{t})$$

and

$$f'(t) = \frac{2\lambda^2}{\sqrt{t}} e^{-(1+t)\lambda} \left( I_1(2\lambda\sqrt{t}) - \sqrt{t} I_0(2\lambda\sqrt{t}) \right)$$

The apparent singularity at  $t = 0$  is removable:  $f$  is analytic at  $t = 0$  with  $f'(0) = 2\lambda^2(\lambda - 1)e^{-\lambda}$  and  $f''(0) = \lambda^3(\lambda^2 - 4\lambda + 2)e^{-\lambda}$ .

Thus for  $0 < \lambda \leq 1$ ,  $t = 0$  is a local maximum of  $f(t)$  on  $[0, 1]$ , while for  $\lambda > 1$  it is a local minimum.

The other endpoint,  $t = 1$ , is never a local maximum. We have

$$f'(1) = -2\lambda^2 e^{-2\lambda} (I_0(2\lambda) - I_1(2\lambda))$$

The fact that  $I_0(2\lambda) > I_1(2\lambda)$  for all  $\lambda > 0$ , and thus that  $f'(1) < 0$ , can be seen using series expansions and the Cauchy-Schwarz inequality: if  $a_k = \lambda^k/k!$ ,

$$\begin{aligned} I_0(2\lambda) &= \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(k!)^2} = \sum_{k=0}^{\infty} a_k^2 \\ I_1(2\lambda) &= \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{k!(k+1)!} = \sum_{k=0}^{\infty} a_k a_{k+1} \end{aligned}$$

But by Cauchy-Schwarz,

$$\sum_{k=0}^{\infty} a_k a_{k+1} \leq \left( \sum_{k=0}^{\infty} a_k^2 \right)^{1/2} \left( \sum_{k=0}^{\infty} a_{k+1}^2 \right)^{1/2} \leq \sum_{k=0}^{\infty} a_k^2$$

Now we investigate critical points in  $(0, 1)$ .

At a critical point of  $f(t)$  we will have  $I_1(2\lambda\sqrt{t})/I_0(2\lambda\sqrt{t}) = \sqrt{t}$ . It is useful to make the substitution  $s = 2\lambda\sqrt{t}$ , so this equation can be written as  $R(s) = 1/(2\lambda)$  where  $R(s) = I_1(s)/(sI_0(s))$ . I claim that  $R(s)$  is a decreasing function of  $s > 0$ , with

$$\lim_{s \rightarrow 0^+} R(s) = 1/2 \quad \text{and} \quad \lim_{s \rightarrow \infty} R(s) = 0$$

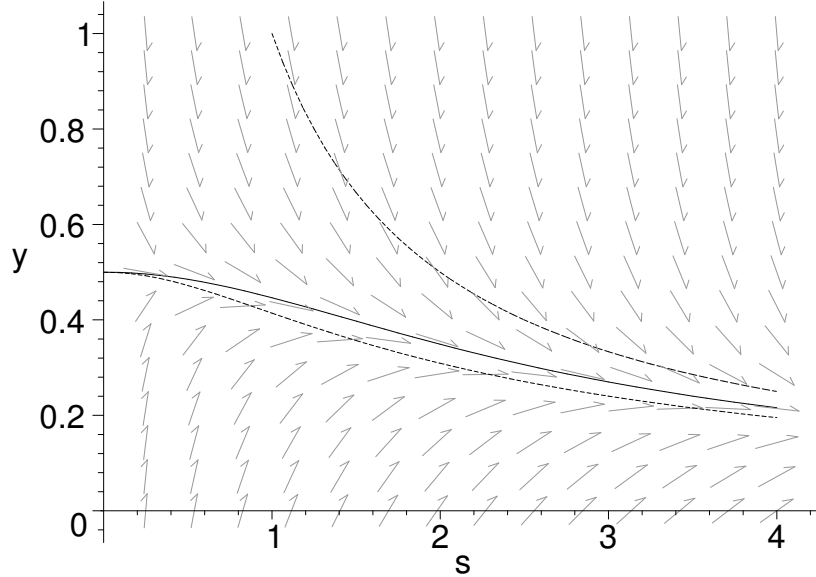
Thus when  $\lambda \leq 1$  there is no positive critical point of  $f(t)$ , while when  $\lambda > 1$  there is exactly one positive critical point of  $f(t)$ . Moreover, since we know  $f'(1) < 0$ , this critical point satisfies  $t < 1$ .

To prove the claim, we use a phase-plane analysis of the differential equation

$$y' = (1 - 2y)/s - sy^2$$

which is satisfied by  $y = R(s)$ .

Here is a phase-plane plot for the differential equation in the first quadrant.



The solid curve is the solution  $y = R(s)$ , while the two dotted curves are the null isocline  $(1 - 2y)/s - sy^2 = 0$  (below) and  $y = 1/s$  (above). In the terminology of Hubbard[1], we will show that the dotted curves are two fences forming a funnel, within which the solution is trapped.

The equation for the null isocline can be solved for  $y$ : the positive solution is

$$y = \frac{\sqrt{1 + s^2} - 1}{s^2}$$

The right side of this is a decreasing function of  $s > 0$ , since its derivative can be written as

$$-\frac{(\sqrt{1+s^2}-1)^2}{s^3\sqrt{1+s^2}}$$

Since the slope (namely 0) of solutions of the differential equation is greater than the slope of the null isocline at all points on that curve, this is a lower fence: solution curves can cross the null isocline from below to above, but not vice versa.

On the other hand, the curve  $y = 1/s$  has slope  $-1/s^2$ , while on that curve the solutions of the differential equation have slope  $(1 - 2/s)/s - s/s^2 = -2/s^2$ . Since the slope of the solutions is less than the slope of the curve at all points on the curve, this is an upper fence: solution curves can cross  $y = 1/s$  from above to below, but not vice versa.

As  $s \rightarrow 0+$  we have  $I_0(s) \rightarrow 1$  and  $I_1(s)/s \rightarrow I_1'(0) = 1/2$ . The solution  $R(s)$  that we are interested in starts (as  $s \rightarrow 0+$ ) on the lower fence, and therefore can't reach any point below that fence, nor can it cross the upper fence. Thus it is trapped between the fences for all  $s > 0$ . Since  $R(s)$  is above the null isocline,  $R'(s) < 0$ ; since it is below the upper fence,  $R(s) \rightarrow 0$  as  $s \rightarrow \infty$ .  $\square$

We mention in passing that the function  $R(s)$  has an interesting continued-fraction representation. This comes from the recursion

$$I_n(s) = I_{n+2}(s) + \frac{2n+2}{s}I_{n+1}(s)$$

which can be written as

$$\frac{sI_{n+1}(s)}{I_n(s)} = \frac{s^2}{2n+2 + \frac{sI_{n+2}(s)}{I_{n+1}(s)}}$$

Thus

$$\begin{aligned} R(s) = \frac{I_1(s)}{sI_0(s)} &= \frac{1}{2 + \frac{sI_2(s)}{I_1(s)}} = \frac{1}{2 + \frac{s^2}{4 + \frac{sI_3(s)}{I_2(s)}}} \\ &= \frac{1}{2 + \frac{s^2}{4 + \frac{s^2}{6 + \frac{s^2}{8 + \frac{s^2}{\dots}}}}} \end{aligned}$$

While there is probably no closed-form expression for the location  $t_{max}$  of the maximum when  $\lambda > 1$ , series and asymptotics are available. Thus by reversion of power series we can obtain a series in powers of  $\lambda - 1$ :

$$t_{max} = 2(\lambda-1) - \frac{10}{3}(\lambda-1)^2 + \frac{43}{9}(\lambda-1)^3 - \frac{841}{135}(\lambda-1)^4 + \frac{6217}{810}(\lambda-1)^5 - \frac{51713}{5670}(\lambda-1)^6 + \dots$$

We can also obtain a series in inverse powers of  $\lambda$ , useful as  $\lambda \rightarrow \infty$ :

$$t_{max} = 1 - \frac{1}{2\lambda} - \frac{1}{8\lambda^2} - \frac{3}{32\lambda^3} - \frac{3}{32\lambda^4} - \frac{57}{512\lambda^5} - \frac{309}{2048\lambda^6} - \frac{471}{2048\lambda^7} - \frac{201}{512\lambda^8} + \dots$$

Numerical evidence indicates that the absolute error in this series (truncated to the terms shown) is less than 0.00075 for  $\lambda > 1.8$ .

#### 4. UNEVENLY MATCHED TEAMS

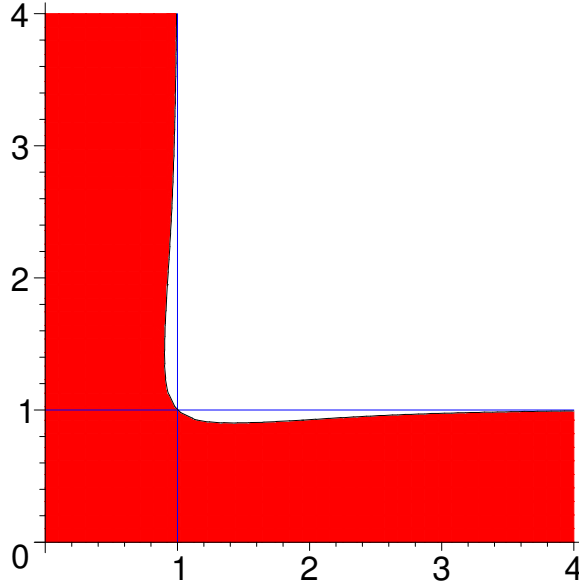
Now consider the general case  $\lambda_1 \neq \lambda_2$ . We have

$$f'(t) = \sqrt{\frac{\lambda_1 \lambda_2}{t}} (\lambda_1 \exp(-\lambda_1 t - \lambda_2) + \lambda_2 \exp(-\lambda_1 - \lambda_2 t)) I_1(2\sqrt{\lambda_1 \lambda_2 t}) \\ - (\lambda_1^2 \exp(-\lambda_1 t - \lambda_2) + \lambda_2^2 \exp(-\lambda_1 - \lambda_2 t)) I_0(2\sqrt{\lambda_1 \lambda_2 t})$$

Again, the apparent singularity at  $t = 0$  is removable, and

$$f'(0) = \lambda_2^2(\lambda_1 - 1)e^{-\lambda_1} + \lambda_1^2(\lambda_2 - 1)e^{-\lambda_2} = \lambda_1^2 \lambda_2^2 \left( \frac{\lambda_1 - 1}{\lambda_1^2} e^{-\lambda_1} + \frac{\lambda_2 - 1}{\lambda_2^2} e^{-\lambda_2} \right)$$

The region of the  $\lambda_1$ - $\lambda_2$  plane where  $f'(0) < 0$ , and thus  $t = 0$  is at least a local maximum of the density, is shown in the following Maple plot.



Writing  $g(x) = (x - 1) \exp(-x)/x^2$ , this region corresponds to the inequality  $g(\lambda_1) + g(\lambda_2) < 0$ . Since  $g$  is negative on  $(0, 1)$  and positive on  $(1, \infty)$ , the region contains all of  $(0, 1] \times (0, 1)$ . We have

$$g'(x) = \frac{2 - x^2}{x^3} e^{-x}$$

so  $g$  is increasing on  $(0, \sqrt{2}]$  and decreasing on  $[\sqrt{2}, \infty)$ . The maximum is at  $x = \sqrt{2}$ . Thus for  $\lambda_1 > 1$  the boundary of the region is the curve  $\lambda_2 = h(\lambda_1)$  where  $h(\lambda_1)$  is the unique solution of  $g(h(\lambda_1)) + g(\lambda_1) = 0$ . For  $1 < \lambda_1 \leq \sqrt{2}$ ,  $h(\lambda_1)$  decreases from 1 to approximately 0.8997630909. For  $\sqrt{2} \leq \lambda_1 < \infty$  it increases; as  $\lambda_1 \rightarrow \infty$  we have  $g(\lambda_1) \rightarrow 0$  so  $h(\lambda_1) \rightarrow 1$ .

Next we show that  $f'(1) < 0$ , so that  $t = 1$  is never a local maximum. Again using the inequality  $I_0(x) > I_1(x)$  for  $x > 0$ , we have

$$\begin{aligned}
f'(1) &= e^{-\lambda_1 - \lambda_2} (\sqrt{\lambda_1 \lambda_2} (\lambda_1 + \lambda_2) I_1(2\sqrt{\lambda_1 \lambda_2}) - (\lambda_1^2 + \lambda_2^2) I_0(2\sqrt{\lambda_1 \lambda_2})) \\
&< e^{-\lambda_1 - \lambda_2} I_0(2\sqrt{\lambda_1 \lambda_2}) (\sqrt{\lambda_1 \lambda_2} (\lambda_1 + \lambda_2) - (\lambda_1^2 + \lambda_2^2)) \\
&= -e^{-\lambda_1 - \lambda_2} I_0(2\sqrt{\lambda_1 \lambda_2}) (\lambda_1 + \sqrt{\lambda_1 \lambda_2} + \lambda_2) (\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 \leq 0
\end{aligned}$$

For  $0 < t < 1$ , we can write the equation  $f'(t) = 0$  as

$$R(2\sqrt{\lambda_1 \lambda_2} t) = \frac{\lambda_1^2 e^{\lambda_1(1-t)} + \lambda_2^2 e^{\lambda_2(1-t)}}{\lambda_2 \lambda_1^2 e^{\lambda_1(1-t)} + \lambda_1 \lambda_2^2 e^{\lambda_2(1-t)}} = \frac{A(\lambda_1, t) + A(\lambda_2, t)}{\lambda_2 A(\lambda_1, t) + \lambda_1 A(\lambda_2, t)}$$

where  $A(\lambda, t) = \lambda^2 \exp(\lambda(1-t))$ .

It seems likely that the behaviour is similar to that of the the case  $\lambda_1 = \lambda_2$  in that there is a single maximum in  $(0, 1)$  whenever 0 is not a local maximum, but we have not been able to prove this.

#### REFERENCES

- [1] HUBBARD, J.H. AND WEST, B. (1991) *Differential Equations, A Dynamical Systems Approach Part I*, Texts in Applied Mathematics No. 5, Springer-Verlag, New York.

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