Homework 2

Math 615

WDVV for $\mathbb{P}^1 \times \mathbb{P}^1$

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ and consider the basis for $H^*(S)$ given by

$$T_0 = 1, \quad T_1 = (\mathbb{P}^1 \times \{pt\})^\vee, \quad T_2 = (\{pt\} \times \mathbb{P}^1)^\vee, \quad T_3 = pt^\vee$$

and let $t_0, t_1, t_2, t_3, q_1, q_2$ be the corresponding variables for the genus 0 Gromov-Witten potential. Let N_{d_1,d_2} be the number of rational curves of bidegree (d_1, d_2) on S passing through the appropriate number of fixed points (you should compute that number). Let $\gamma = \sum_{i=0}^{3} t_i T_i$ and recall that the genus zero Gromov-Witten potential is defined by

$$F = \sum_{\beta} \langle \exp(\gamma) \rangle_{0,\beta} \, q^{\beta}.$$

1. Explicitly write out the potential function F in terms of the N_{d_1,d_2}'s.
2. Use the WDVV equation $F_{\alpha\beta\epsilon} g^{\epsilon\epsilon'} F_{\epsilon'\gamma\delta} = F_{\alpha\gamma\epsilon} g^{\epsilon\epsilon'} F_{\epsilon'\beta\delta}$ with $(\alpha\beta\gamma\delta) = (1233)$ to derive a recursive formula for N_{d_1,d_2}.
3. How many rational curves of bidegree (2,2) pass through 7 generic points? For what degree d does $N_{2,2}$ equal N_d, the number of rational plane curves of degree d passing through $3d - 1$ points? Can you give a geometric explanation for this equality?