Homework 2b

Math 615

February 22, 2024

Problem 1, Connected vs disconnected invariants

Let X be a Calabi-Yau threefold and let $\overline{\mathcal{M}}_g(X,\beta)$ be the moduli space of (connected) stable maps of genus g and degree β, and let $\overline{\mathcal{M}}^{\bullet}_{\chi}(X,\beta)$ be the moduli space of possibly disconnected stable maps of degree β and where the domain curve C has $\chi(\mathcal{O}_C)=\chi$. Let $N_{g,\beta}$ and $N^{\bullet}_{\chi,\beta}$ be the corresponding connected and disconnected Gromov-Witten invariants. Let F and Z be the potential function and the partition function:

$$F=\sum_{g,\beta} N_{g,\beta}\lambda^{2g-2}v^{\beta}$$

$$Z=\exp\left(F\right).$$

Show that Z is the generating function for the disconnected invariants, namely:

$$Z=\sum_{\chi,\beta} N^{\bullet}_{\chi,\beta}\lambda^{-2\chi}v^{\beta}.$$

You may assume the following reasonable facts about the behaviour of the degree of the virtual class under disjoint union, products, and quotients by a finite group.

1. If $M=M_1\sqcup M_2$ then $\deg[M]^{vir}=\deg[M_1]^{vir}+\deg[M_2]^{vir}$.
2. If $M=M_1\times M_2$ then $\deg[M]^{vir}=\deg[M_1]^{vir}\cdot\deg[M_2]^{vir}$.
3. If $M_1=M_2/G$ where G is a finite group of order n, then $\deg[M_1]^{vir}=\deg[M_2]^{vir}/n$.

Problem 2, Unramified covers of the torus.

Let E be a smooth genus 1 projective curve. We wish to compute the Gromov-Witten invariants $N_{1,d[E]}(E)$. As we showed in class, all stable maps in $\overline{\mathcal{M}}_1(E, d[E])$ consist of unramified covers $f : F \to E$ by a connected genus 1 curve F.

Part 1. Prove that

$$N_{1,d[E]}(E) = \frac{1}{d}\sigma(d) = \frac{1}{d}\sum_{k|d} k$$

by showing that the number of (connected) covering spaces is given by $\sigma(d)$ and that each such space is a normal covering space with a group of deck transformations of order d.

The *disconnected* Gromov-Witten invariant are given by

$$N^{\bullet}_{\chi=0,d[E]}(E) = p(d)$$

where $p(d)$ is the number of partitions of the integer d. This formula follows from the formula for the connected invariants by using problem 1, or can be computed directly by counting disconnected unramified covers.

A (possibily disconnected), degree d, unramified cover $f : F \to E$ is determined by its monodromy: fixing a set isomorphism $f^{-1}(x_0) \cong \{1, \ldots, d\}$, we get a permutation for every loop in E beginning and ending at $x_0 \in E$ by lifting paths to the cover. In this way we get a homomorphism

$$\pi_1(E, x_0) \to S_d$$

which uniquely determines the cover, upto the choice of the isomorphism $f^{-1}(x_0) \cong \{1, \ldots, d\}$.

Part 2. Show that

$$\frac{1}{d!} \cdot |\operatorname{Hom}(\pi_1(E, x_0), S_d)| = p(d).$$

By the previous discussion, the left hand side counts the number of degree d unramified covers of E. The factor $\frac{1}{d!}$ undoes the extraneous choice of the isomorphism $f^{-1}(x_0) \cong \{1, \ldots, d\}$ and correctly counts each cover by the reciprocal of the number of its automorphism.

Problem 3, Inverting the Gopakumar-Vafa formula.

Recall that the Gopakumar-Vafa formula gives the following relationship between $\{N_{g,\beta}(X)\}$, the Gromov-Witten invariants of a CY3 X, and $\{n_{g,\beta}(X)\}$, the Gopakumar-Vafa invariants of X:

$$\sum_{\beta\neq 0}\sum_{g\geq 0} N_{g,\beta}(X)\lambda^{2g-2}v^{\beta} = \sum_{\beta\neq 0}\sum_{g\geq 0} n_{g,\beta}(X)\sum_{k>0}\frac{1}{k}\left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2} v^{k\beta}.$$

Suppose that $\beta \in H_2(X,\mathbb{Z})$ is a primitive curve class.

1. Write the GW invariant $N_{1,4\beta}(X)$ as a linear combination of the GV invariants $n_{0,\beta}(X)$, $n_{0,2\beta}(X)$, $n_{0,4\beta}(X)$, $n_{1,\beta}(X)$, $n_{1,2\beta}(X)$, and $n_{1,4\beta}(X)$.
2. Write the GV invariant $n_{1,4\beta}(X)$ as a linear combination of the GW invariants $N_{0,\beta}(X)$, $N_{0,2\beta}(X)$, $N_{0,4\beta}(X)$, $N_{1,\beta}(X)$, $N_{1,2\beta}(X)$, and $N_{1,4\beta}(X)$.