Lecture 1

Enumerative Geometry Beyond Numbers

[mention some prerequisites: bundles, sheaf cohomology, chern classes, some schemes]

This course is an introduction to modern enumerative algebraic geometry. Classically, enumerative geometry is about counting problems in algebraic geometry, for example we may ask for the number a curves in some space satisfying some set of conditions. Some classical examples include

- How many lines are there on a smooth cubic surface $S \subset \mathbb{CP}^3$?

  A: 27 (Cayley & Salmon 1849)

- How many lines are bitangent to a smooth degree $d > 1$ curve $C \subset \mathbb{CP}^2$?

  A: $\frac{1}{2} d(d-2)(d-3)(d+3)$ (Plucker 1830's)

Recall that the geometric genus of a (possibly singular) curve C is the usual genus of the normalization $\tilde{C} \to C$. A curve is called rational if it has geometric genus 0.

- Let N_d = # of rational plane curves of degree d passing through $3d-1$ general points. e.g.

N_1 = # of lines through 2 points = 1
N_2 = # of conics through 5 points = 1
N_3 = # of rational cubics through 8 points = 12
N_4 = # of rational quartics through 11 points = 620

} Greeks

easy with classical AG

hard with classical AG 1800s

First "non-classical" result. Kontsevich 1994:

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_i>0}}^{\prime} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

The above was very striking not just because it gives a complete solution to a difficult classical problem, but because it is a corollary of the fact that the product in $QH^*(\mathbb{P}^2)$, the <u>Quantum Cohomology</u> of $\mathbb{P}^2$, is associative. Quantum cohomology is an idea that comes from the physics of string theory and is a deformation of the usual cohomology ring.

<u>Modern Enumerative Geometry</u> 1990's-present. Ideas coming from physics (mainly string theory but also varius quantum field theories) have driven new developments in enumerative algebraic geometry. The focus is not so much on the actual numbers, but on new structures arising from the numbers ("beyond numbers").

Another extremely influential example from the 1990s:

Let $X^3_{(5)} \subset \mathbb{CP}^4$ be a generic quintic threefold (an example of a CY3).

Let n_d be the number of degree d rational curves on $X^3_{(5)}$.

$n_1 = 2875$, $n_2 = 609250$ (Katz 1986)

Using mirror symmetry in string theory Candelas, dela Ossa, Green conjectured (1991) a general formula for these numbers.

Lecture 2 Calabi-Yau threefolds play a central role in this story. Most of what we will study in this course are quantum invariants of Calabi-Yau threefolds.

Def'n: A Calabi-Yau manifold of dimension n is a non-singular complex projective variety $X \subset \mathbb{CP}^N$ with $\dim X = n$ and $H^2(X)_{tor} = 0$, satisfying one of the following equivalent conditions:

① X admits a Kähler metric whose Ricci curvature is zero (called a Ricci-flat or Calabi-Yau metric).

② X admits a non-vanishing holomorphic n-form

③ The canonical line bundle $K_X = \Lambda^n T^*_X$ is trivial: $K_X \cong \mathcal{O}_X$.

Remarks:

- ① $\Rightarrow$ ② $\Leftrightarrow$ ③ is easy. ② $\Rightarrow$ ① is Yau's fields medal theorem
- As algebraic geometers, we will often take ③ as the definition, even in the case where X is not compact (only quasi-projective). in that case ② $\not\Rightarrow$ ① and ① may fail, but we still consider $K_X \cong \mathcal{O}_X$ to be CY.
- Some people require $H^k(X, \mathcal{O}_X) = \begin{cases} \mathbb{C} & k=0,n \\ 0 & \text{otherwise} \end{cases}$

Examples:

$\dim_{\mathbb{C}} X = 1$ CY3s are elliptic curves, a.k.a. genus 1 Riemann surfaces, a.k.a. smooth cubic plane curves $X_{(3)} \subset \mathbb{P}^2$. There is only one topological type

$\dim_{\mathbb{C}} X = 2$ CY3s are either Abelian surfaces $A \cong \mathbb{C}^2/\mathbb{Z}^4$ or K3 surfaces. An example of a K3 surface is a smooth quartic surface in $\mathbb{P}^3$ $X_{(4)} \subset \mathbb{P}^3$. There are two topological types.

$\dim_{\mathbb{C}} X = 3$ Has a vast number of distinct topological types (possibly infinite, conjecturally finite, probably $> 500{,}000{,}000$). For example a smooth quintic hypersurface $X_{(5)} \subset \mathbb{CP}^4$

In general a degree $N+1$ hypersurface in $\mathbb{CP}^N$ is a CY $(N-1)$ fold

$X_{(N+1)} \subset \mathbb{CP}^N$ $\quad$ $X_{(N+1)} = \{ (x_0 : \cdots : x_N) : F(x_0, \cdots, x_N) = 0 \;\; F$ homogeneous poly in $N+1$ variables of degree $N+1 \}$

Generalizing this example:

The zero locus of a section of the dual canonical bundle is CY:

If M is any cx projective manifold of dim $N+1$ then $s^{-1}(0)$ is a CYn where

$$\begin{array}{c} K_M^\vee \\ s \curvearrowright \big\downarrow \\ M \supset s^{-1}(0) \end{array} \qquad \left(\begin{array}{c} \text{if } M = \mathbb{CP}^{N+1} \text{ then } K = \mathcal{O}(-N-1) \text{ so } K^\vee = \mathcal{O}(N+1) \text{ so} \\ \text{sections are degree } N+1 \text{ polynomials} \end{array} \right)$$

more generally, if $E \to M$ is a rk r vector bundle with $\Lambda^r E \cong K_M^\vee$ and $s : M \to E$ is a section intersecting the zero section transversely, then $s^{-1}(0) \subset M$ is a CY of dim $\dim M - \mathrm{rk} E$.

<u>Non-compact example</u> Let $E \to M$ be a rk r vector bundle with $\Lambda^r E \cong K_M$, then $X = \mathrm{Tot}(E)$ is a CY of dim $\dim M + \mathrm{rk} E$.

example: $M = \mathbb{P}^1 \quad E = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ so $\Lambda^2 E = \mathcal{O}(-2) = K_{\mathbb{P}^1}$

$X = \mathrm{Tot}\left(\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1\right)$ is a CY3

$$= \left\{ (x_0, x_1, v, w) : (x_0, x_1) \neq (0,0) \right\} \Big/ (x_0, x_1, v, w) \sim (\lambda x_0, \lambda x_1, \lambda^{-1} v, \lambda^{-1} w)$$

"local $\mathbb{P}^1$" a.k.a.

"conifold resolution"

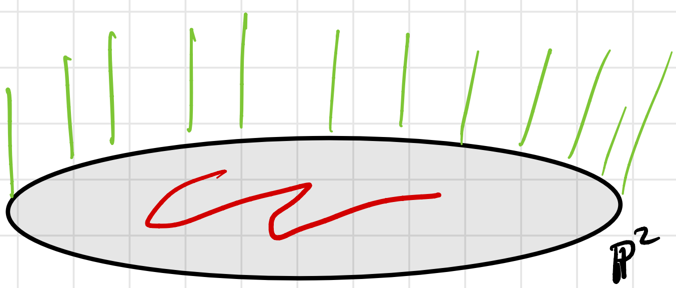
conifold singularity

$$\begin{array}{ccc} X & \longrightarrow & X_{sing} = \{ xy = wz \subset \mathbb{C}^4 \} \\ \cup & & \\ \mathbb{P}^1 & \longmapsto & (0,0,0,0) \end{array}$$

The only projective curve in X is the zero section (why?)

example $M = \mathbb{P}^2$, $E = K_{\mathbb{P}^2} = \mathcal{O}(-3)$, $X = \mathrm{Tot}\left(\mathcal{O}(-3) \to \mathbb{P}^2\right)$ is a CY3 ("local $\mathbb{P}^2$")

$$X = \left\{ (x_0, x_1, x_2, v) : (x_0, x_1, x_2) \neq (0,0,0) \right\} \Big/ (x_0, x_1, x_2, v) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda^{-3} v)$$



Curves in X must lie in $\mathbb{P}^2 \subset X$.

$$X \longrightarrow X_{sing} \cong \mathbb{C}^3/\mathbb{Z}_3 \quad (x,y,z) \sim (\omega x, \omega y, \omega z) \quad \omega = e^{2\pi i/3}$$

Lecture 3

Quantum Invariants of CY3s is a catch all phrase which refers to **deformation invariants** having close ties to or analogs in string theory or quantum field theory.

A **deformation invariant** is a quantity (typically a number) associated to a projective manifold which is invariant under deformations of the complex structure. (Simple example is any topological invariant).

The invariants we study in this class arise from (virtual) counts of curves $C \subset X$. A curve defines a homology class $[C] \in H_2(X, \mathbb{Z})$ and we typically count curves in X having a fixed homology class $\beta \in H_2(X, \mathbb{Z})$ and fixed genus g.

To perform such a count we'd like to define a space (I'm being vague here) $M_g(X, \beta)_{sm}$ which parameterizes (say) smooth curves $C \subset X$ with $[C] = \beta$ and genus g. This is called a moduli space (each point in $M_g(X, \beta)_{sm}$ corresponds to a smooth curve $C \subset X$). In the ideal case, $M_g(X, \beta)_{sm}$ is a finite set of points and then the number of points in the moduli space is the number of genus g curves in the class β. In that case $M_g(X, \beta)_{sm}$ is a **0-dim'l projective manifold**.

If X is a CY3 then the <u>expected dimension</u> of $M_g(X,\beta)_{sm}$ is 0 for any β and g. This is one reason why CY3s are very special.

<u>Expected dimension</u>: If a variety M is given by the zero locus of a set of equations on an ambient manifold W, then the <u>expected dimension</u> of M is $\dim W - \#$ of equations. If the solution sets of the equations intersect transversely, then M is smooth and of the expected dimension, otherwise M may be singular and/or have dimension larger than the expected dimension. (Jargon: M arises from <u>excess intersection</u>).

For X a CY3, often $M_g(X,\beta)_{sm}$ has dimension >0 and/or is singular. We would still like to use $M_g(X,\beta)_{sm}$ to obtain a numerical invariant, a "virtual" count of curves in X of genus g and class β. To do this we must surmount two fundamental problems:

- $M_g(X,\beta)_{sm}$ is (typically) non-compact. We must <u>compactify</u> the space of smooth curves.

- $M_g(X,\beta)_{sm}$ is (typically) singular and has excess dimension due to non-transversality of defining equations of $M(X,\beta)_{sm}$.

Different approaches to resolving these issues leads to different kinds of invariants. The basic blueprint is:

$$\underset{CY3}{X} \rightsquigarrow M(X,\beta,g) \rightsquigarrow \text{Numerical deformation invariant}$$

Some compact moduli space of curves in the class β and genus g

Usually the degree of a "virtual class" sometimes something more exotic like a rank of a chomology group. This process should account for M having access dimension and/or being singular.

The two basic strategies:

- curves are parameterized, they are given by maps $f: C \to X$.

$\rightsquigarrow$ moduli space of stable maps $\rightsquigarrow$ GW invariants (world sheets in string theory).

- curves are cut out by equations, they are given by sheaves.

$\rightsquigarrow$ varius moduli spaces of sheaves,

Ideal sheaves / Hilbert scheme $\rightsquigarrow$ DT invariants

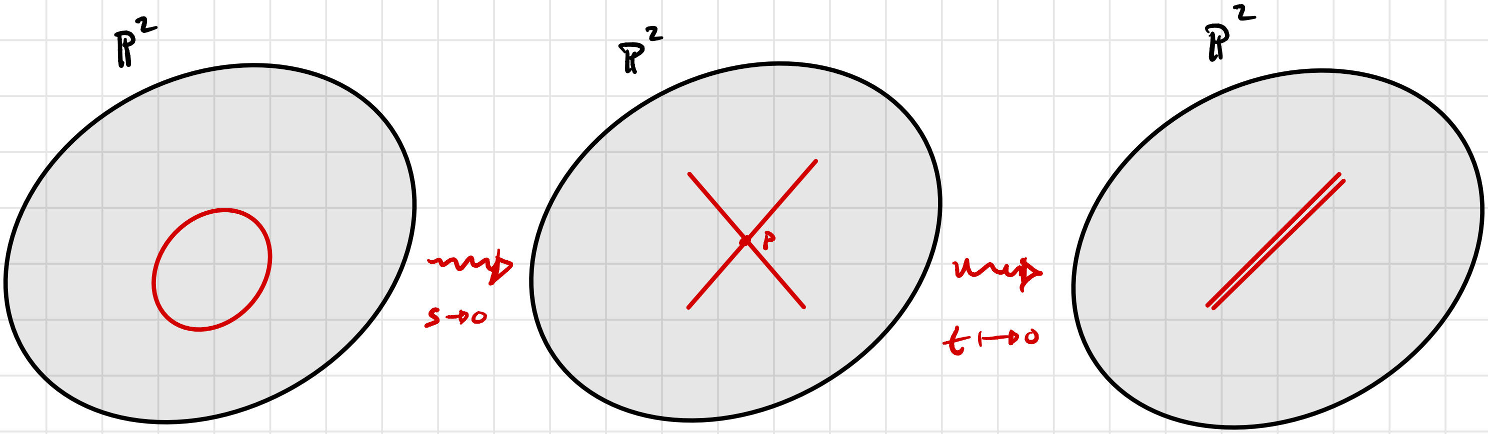
Torsion sheaves with section (stable pairs) $\rightsquigarrow$ PT invariants

torsion sheaves $\rightsquigarrow$ MT/GV invariants

(these are varius D-branes in string theory)

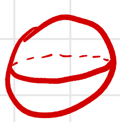
In a family of curves in a projective manifold X, a smooth curve can degenerate to a singular curve:

<u>Examples</u> Conic curves in $\mathbb{P}^2$ $\leftarrow$ coords $(x:y:z)$



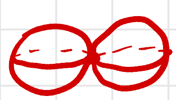
$(x+ty)x+sz=0$

smooth conic is a $\mathbb{P}^1$ in the class $2[L] \in H_2(\mathbb{P}^2, \mathbb{Z})$



$(x+ty)x=0$

pair of lines $\mathbb{P}^1 \cup_p \mathbb{P}^1$ having a <u>node</u> as a singularity



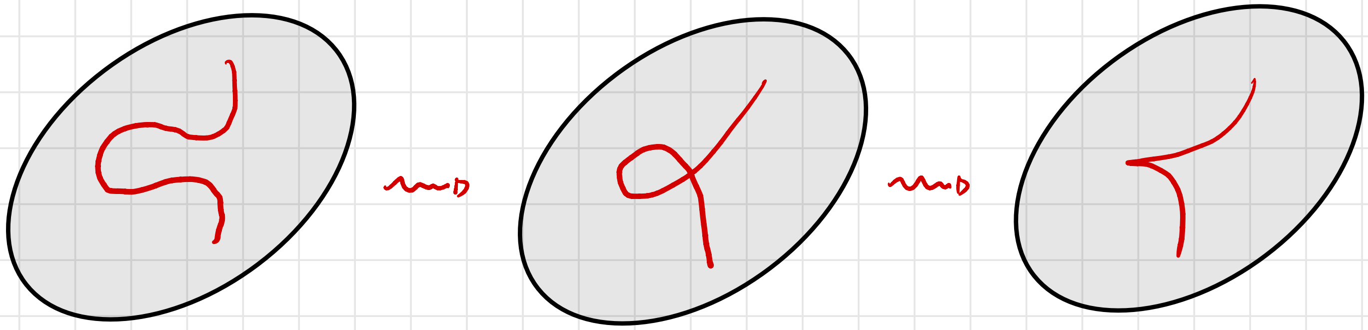
$x^2=0$

"doubled" line

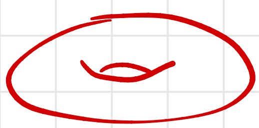
same locus as $x=0$, but we want something that reflects the fact that it came from a conic.

<u>Lecture 4</u>

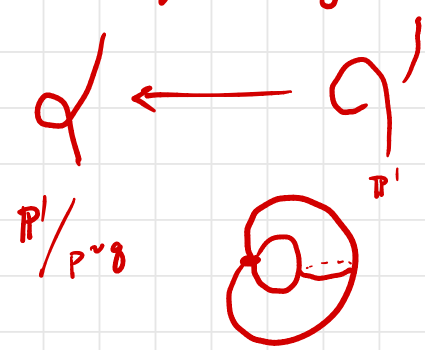
example: cubic curves in $\mathbb{P}^2$



smooth cubic curve genus 1 in $3[L] \in H_2(\mathbb{P}^2)$

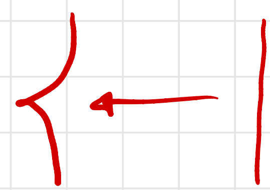


nodal cubic, it has arithmetic genus 1 and geometric genus 0



node is locally $xy=0$
(analytically, or formally)

cuspidal cubic affine equation $x^2=y^3$, normalization is $\mathbb{P}^1$ and is bijective



Different moduli spaces handle these degenerations differently which leads to different curve counting theories.

Gromov-Witten Theory We consider curves in X as given by their embedding map $f: C \to X$. We allow C to have singularities, but only nodes, but we no longer require f to be an embedding.

Def'n A **stable map** to X of genus g and class $\beta \in H_2(X, \mathbb{Z})$ is a map $f: C \to X$ where C is a connected curve of (arithmetic) genus g with at worst nodal singularities, $f_*[C] = \beta$, and such that

$$\mathrm{Aut}(f: C \to X) = \{\phi \in \mathrm{Aut}(C) : f \circ \phi = f\}$$ is finite.

Def'n Two stable maps $f: C \to X$, $f': C' \to X$ are **equivalent** if there exists $\phi: C \to C'$ isomorphism such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\phi"', "\cong"] & X \\ C' \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes.

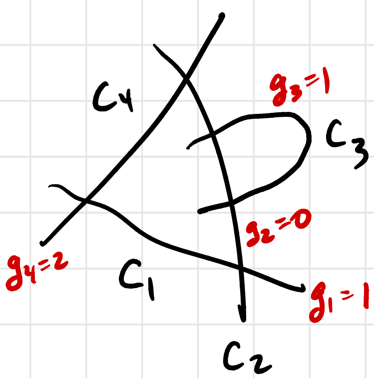
Theorem If X is projective, then the moduli space of stable maps $\overline{M}_g(X, \beta)$ is compact [it is a projective Deligne-Mumford stack]. There is a projective variety whose points correspond bijectively with equivalence classes of stable maps. (Every flat family of stable maps induces a morphism).

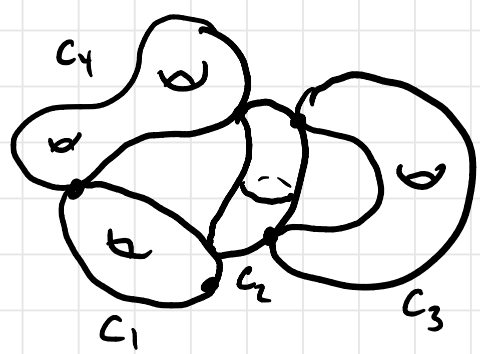
Special Case of theorem is $X = pt$. $\overline{M}_g(pt, 0) = \overline{M}_g$ $(g > 1)$ ← why?

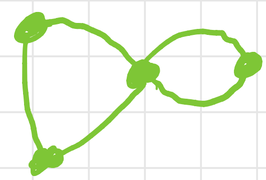
Deligne-Mumford moduli space of stable curves. Smooth orbifold of dimension $3g-3$. It is a compactification of M_g, the moduli space of smooth curves.

The arithmetic genus of a connected curve C is by definition $\operatorname{rk} H^1(C, \mathcal{O})$,

for a nodal curve $C = \bigcup_i C_i$ $\quad g = \sum_i g(C_i) + \underbrace{1 - e(\Gamma)}_{\#\text{ of cycles in dual graph}}$

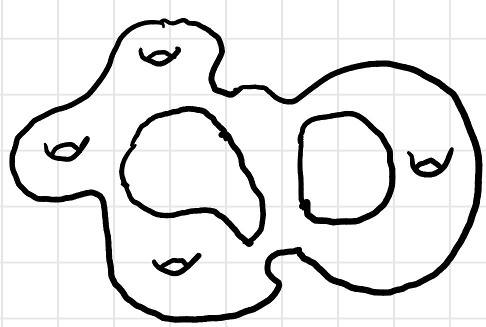




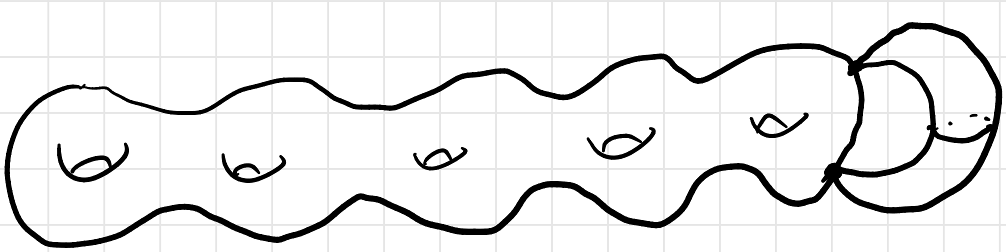


dual graph has a node for each C_i and an edge for each node

smooths to



$g = 6$



← genus 6 but unstable

A nodal curve has finite automorphism group iff every rational component has 3 or more nodes, every elliptic component has at least 1 node ($\overline{m}_1 = \phi$).

Lecture 5

For stable maps $f: C = \bigcup_i C_i \longrightarrow X$ each $f_i = f|_{C_i}$ has some "degree" $f_{i*}[C_i] = \beta_i \in H_2(X, \mathbb{Z})$ if $\beta_i \neq 0$ there are only a finite # of automorphisms of f that can act non-trivially on C_i : they can only permute points $\{f^{-1}(pt)\}$

$|Aut(f: C \to X)| = \infty \Rightarrow \exists$ some component C_i of degree 0 such that $|Aut(C_i, \text{nodes})| = \infty$ i.e. $C_i = \mathbb{P}^1$ with 2 or fewer nodes or $g(C_i) = 1$ with no nodes.

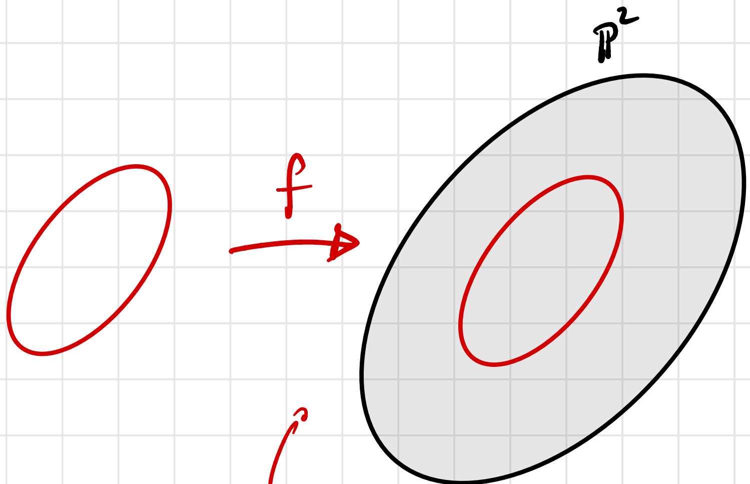
> Stability $\Leftrightarrow$ Every genus 0 collapsing component must have 3 or more nodes (and $\overline{m}_1(X, 0) = \phi$).

Let's see what happens in the examples

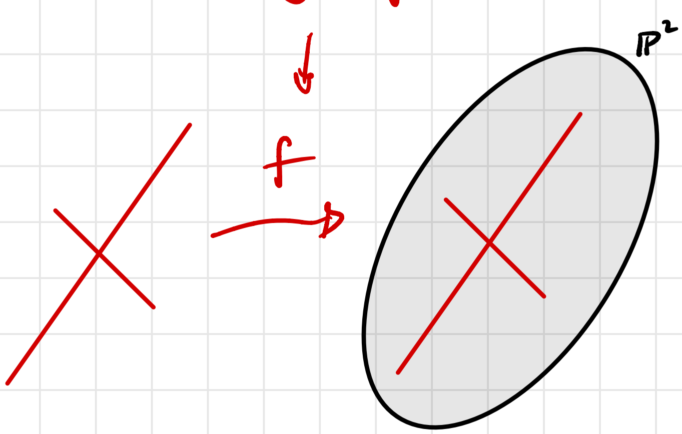
$$\begin{tikzcd} \overline{M}_0(\mathbb{P}^2, 2[L]) \arrow[r] & \mathbb{P}^5 \\ \text{stable maps} \arrow[u] & \text{divisors} \arrow[u] \end{tikzcd}$$

$\leftarrow$ linear system of conics

$$\mathbb{P}^5 = (a:b:c:d:e:f)$$
$$\updownarrow$$
$$ax^2+by^2+cz^2+dxy+eyz+fxz=0$$



embedding map is the stable map



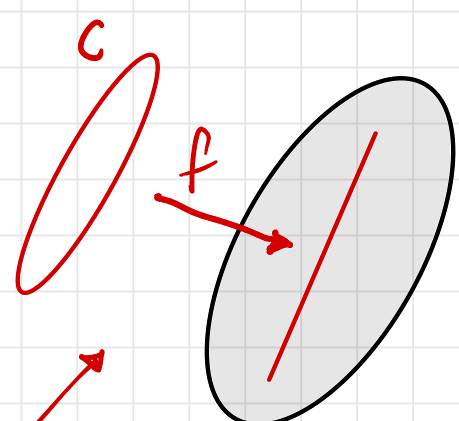
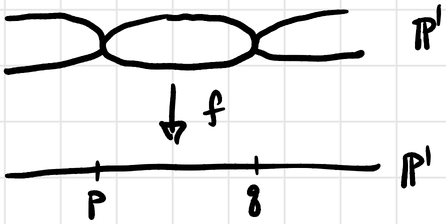


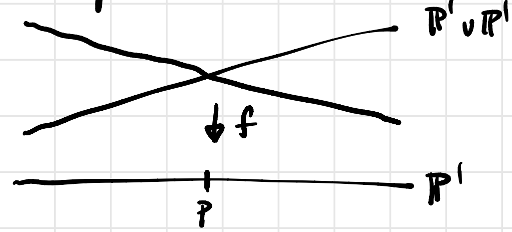
Image is a line but map must be 2:1 cover since $f_*[C] = 2[L]$.

Double covers $f: \mathbb{P}^1 \to \mathbb{P}^1$ have two branch points and the map is determined by those points:



stable map has a $\mathbb{Z}/_2$ automorphism.

If p and q come together, we still get a stable map:



So set of stable maps which double cover a given line $L \subset \mathbb{P}^2$ is given by $\mathrm{Sym}^2 L = \mathrm{Sym}^2 \mathbb{P}^1 = \mathbb{P}^2$.

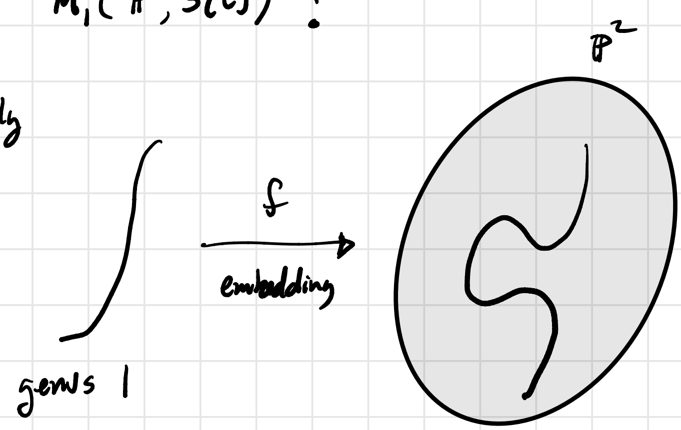
So the map $\overline{M}_0(\mathbb{P}^2, 2[L]) \xrightarrow{\pi} \mathbb{P}^5$ is 1:1 away from the locus of double lines which is $\mathbb{P}^2 \subset \mathbb{P}^5$ embedded by the Veronese embedding and $\pi^{-1}(p) = \mathbb{P}^2$ for any $p \in \mathbb{P}^2 \subset \mathbb{P}^5$

In fact, $\overline{M}_0(\mathbb{P}^2, 2[L]) = Bl_{\mathbb{P}^2}(\mathbb{P}^5)$ [with a $\mathbb{Z}/_2$ orbifold structure along the exceptional divisor]

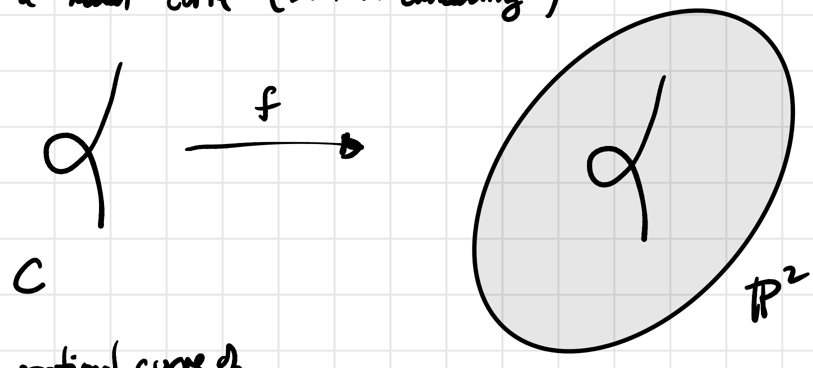
Notice moduli space is smooth here. It is also of the expected dimension.
Note that if $X = \mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$ "local $\mathbb{P}^2$". Then $\overline{M}_0(X, 2[L]) = \overline{M}_0(\mathbb{P}^2, 2[L])$ is smooth but not of the expected dimension.

What can happen in the moduli space $\overline{M}_1(\mathbb{P}^2, 3[L])$?

Now we have genus 1 curves, generically

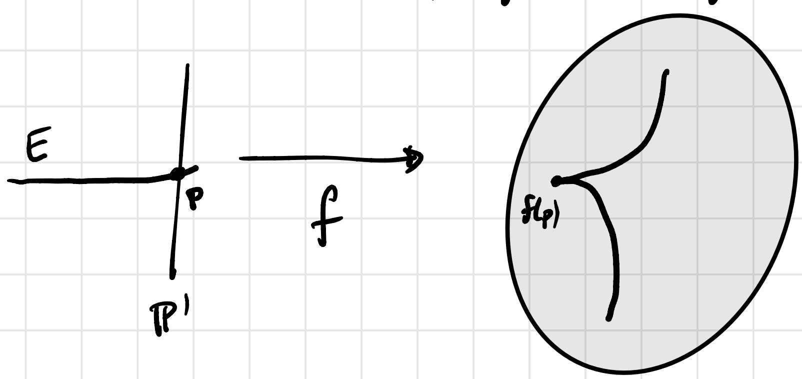


Can degenerate to a nodal curve (still an embedding)



rational curve of arithmetic genus 1

When image is a cuspidal cubic, map can no longer be an embedding.

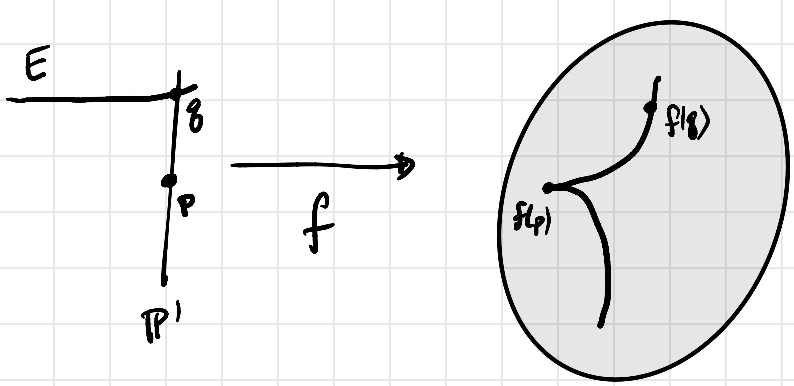


$f : C \longrightarrow \mathbb{P}^2$ $\qquad C = E \cup_p \mathbb{P}^1$ union of elliptic curve and $\mathbb{P}^1$

$f|_{\mathbb{P}^1}$ is the normalization

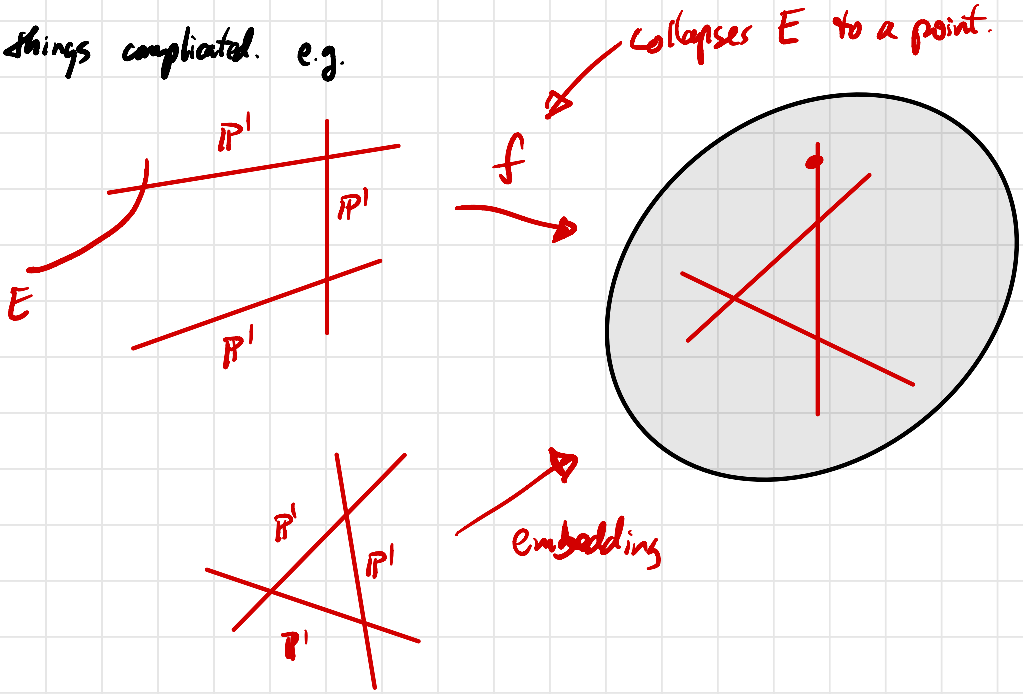
$f|_E$ is a constant map whose image is the cusp.

This map can deform in an interesting way:



This map does not smooth:
there is no infinitesimal deformation of f with a smooth domain $\Rightarrow$ $\overline{M}_1(\mathbb{P}^2, 3[L])$ has multiple irreducible components.

So $\overline{M}_1(\mathbb{P}^2, 3[L])$ is already very complicated. The possibility of collapsing components makes things complicated. e.g.



Homework problems involve seeing what can happen in the moduli spaces

$\overline{M}_2(\mathbb{P}^1, [\mathbb{P}^1])$ & $\overline{M}_1(\mathbb{P}^1, 2[\mathbb{P}^1])$

Lecture 6

Expected dimension (also called virtual dimension)

A central formula in Gromov Witten theory is the following

$$\boxed{\mathrm{virdim}_{\mathbb{C}}(\overline{M}_g(X,\beta)) = -K_X \cdot \beta + (\dim_{\mathbb{C}} X - 3)(1-g)}$$

We will sketch a derivation of this and see how to think about the virtual/expected aspect of the formula, but first some examples:

$$\text{vdim}\,\overline{M}_g(\mathbb{P}^2, d[L]) = (3[L])\cdot(d[L]) + (2-3)(1-g)$$
$$= 3d + g - 1$$

degree d curves forms a linear system $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d))) = \mathbb{P}^{\binom{d+2}{2}-1} = \mathbb{P}^{\frac{1}{2}d(d+3)}$

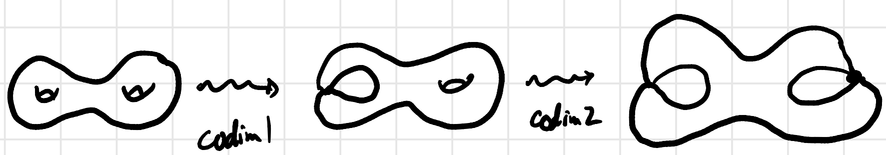
degree d polys in 3 variables

$\bullet\ \bullet | \bullet\ \bullet\ \bullet | \bullet$
d sheep, 2 fences.

a smooth, degree d plane curve has genus $\frac{1}{2}(d-1)(d-2)$ and indeed

if $g = \frac{1}{2}(d-1)(d-2)$ then $3d + \frac{1}{2}(d-1)(d-2) - 1 = \frac{1}{2}d^2 - \frac{3}{2}d + 1 - 1 + 3d = \frac{1}{2}d^2 + \frac{3}{2}d.$

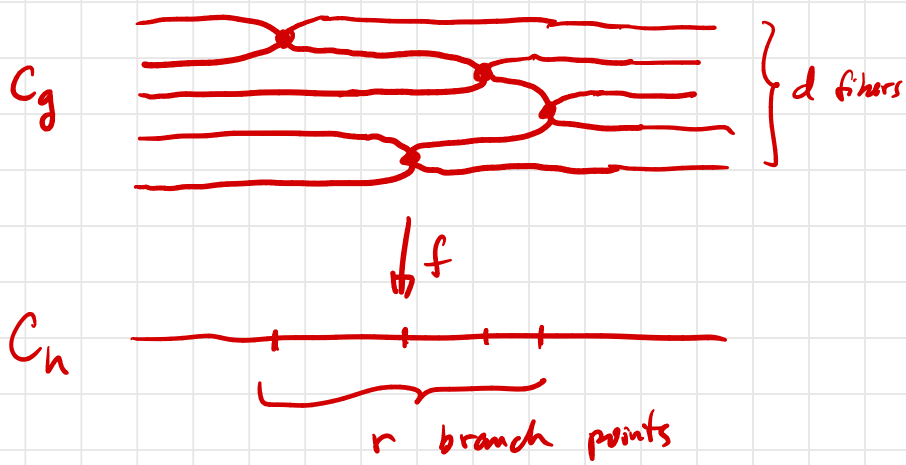
The geometric genus of a curve drops by 1 in codim 1, 2 in codim 2, etc.



$$\text{vdim}\,\overline{M}_g(C_h, d[C_h]) = -K_{C_h}\cdot d[C_h] + (1-3)(1-g) = d(2-2h) + 2g - 2$$
$$= 2g - 2 - d(2h-2)$$

maps $f: C_g \xrightarrow{d:1} C_h$

curve of genus g — fixed curve of genus h



The relationship between g, h, r is given by the Riemann-Hurwitz formula:

$$\underbrace{C_g - (d-1)r \text{ pts}}_{f^{-1}(\text{branch locus})} \longrightarrow \underset{\substack{\uparrow \\ \text{branched locus}}}{C_h - r\text{pts}} \quad \text{is unramified (covering space)}$$

so
$$d\, e(C_h - r\text{pts}) = e(C_g - (d-1)r \text{ pts})$$
$$d(2-2h-r) = 2-2g-(d-1)r$$
$$d(2-2h)-dr = 2-2g-dr+r$$
$$r = 2g-2-d(2h-2)$$

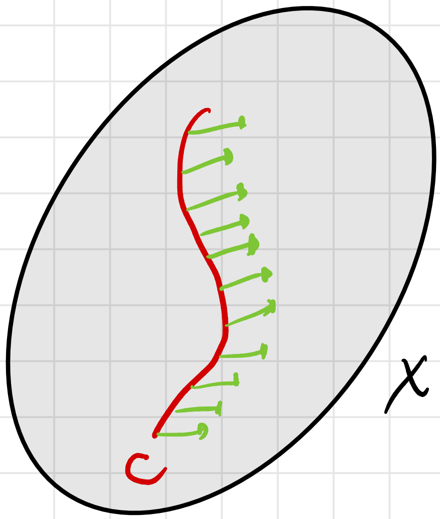
makes sense: only way to deform map is to move location of branched points.

If $\dim X = 3$ $\quad \operatorname{virdim} \overline{M}_g(X,\beta) = -K_X\cdot\beta$ $\quad$ (genus independent)

If X is a CY3 $\quad \operatorname{virdim} \overline{M}_g(X,\beta) = 0$ $\quad$ for all β and g.

Let's understand the dimension formula in the case where $f: C \to X$ is an embedding of a smooth curve. We claim the infinitesimal deformations of $f: C\to X$ are given by $H^0(C, f^*N_{C/X})$ where $N_{C/X}$ is the normal bundle of C in X. $\quad 0 \to T_C \to T_X|_C \to N_{C/X} \to 0$ defines $N_{C/X}$. We get a long exact sequence in cohomology:

$$\begin{array}{ccccccccccccc} 0 \to & H^0(T_C) & \to & H^0(T_X|_C) & \to & H^0(N_{C/X}) & \to & H^1(T_C) & \to & H^1(T_X|_C) & \to & H^1(N) & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \mathrm{Aut}(C) & & \mathrm{Def}(f \text{ fixing } C) & & \mathrm{Def}(f: C\to X) & & \mathrm{Def}(C) & & \mathrm{Ob}(f) & & \mathrm{Ob}(f: C\to X) & \end{array}$$



$$0 \to T_C \to T_X|_C \to N \to 0$$

might not split globally but locally

we may choose a lift $N \to T_X|_C$

and deform. Local lifts will differ on

overlaps by vector fields on C and this data

(Cech 1-cycle valued in vector fields gives rise to an

infinitesimal deformation of C).

we "expect" obstructions to vanish.

$$\operatorname{vdim} \overline{\mathcal{M}}_g(X,\beta) = \dim H^0(C,N) - \dim H^1(C,N)$$

actual dimension

actual dimension = virtual dim when $H^1(C,N) = 0$

by Riemann-Roch $\quad \operatorname{vdim} \overline{\mathcal{M}}_g(X,\beta) = \chi(C,N)$

$$= \deg N + \operatorname{rk} N(1-g)$$

$$= \deg N + (\dim_X - 1)(1-g)$$

and since $\quad 0 \to T_C \to T_X|_C \to N \to 0$

$$\deg N = \deg(T_X|_C) - \deg T_C = -\deg(T_X^*|_C) - (2-2g) = -K_X \cdot C - (2-2g)$$

$$\operatorname{vdim} \overline{\mathcal{M}}_g(X,\beta) = -K_X \cdot \beta - 2(1-g) + (\dim_X - 1)(1-g)$$

$$= -K_X \cdot \beta + (\dim_X - 3)(1-g).$$