Since $\overline{M}_g(C_0, d[C_0]) \subset \overline{M}_g(X, d[C_0])$ is a union of connected components it makes sense to restrict the virtual class

$$N^{GW}_{g+h,d}(C_g \subset X) = \int_{[\overline{M}_{g+h}(X, d[C_g])]^{vir}\big|_{[\overline{M}_{g+h}(C_g, d[C_g])}} 1$$

$$= \int_{[\overline{M}_{g+h}(C_g, d[C_g])]^{vir}} c_D(Ob)$$

virdim $= D = (2-2g)d + 2(g+h) - 2$

$= 2h + (2-2g)(d-1)$

$c_D(Ob)$ is D-th chern class of Obstruction sheaf. Fibers of obstruction sheaf are $H^1(C_{g+h}, f^* N_{C_g/X})$

Lecture 15

Example: if $C_0 \subset X$ has $C_0 \cong \mathbb{P}^1$ and $N_{C_0/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ then C_0 is super rigid.

local $\mathbb{P}^1$, a.k.a. resolved conifold

$$N_{h,d}(C_0 \subset X) = \int_{[\overline{M}_h(\mathbb{P}^1, d[\mathbb{P}^1])]^{vir}} c_D(Ob) \longleftarrow = N_{h,d}\left(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right)$$

This can be computed using the $\mathbb{C}^\times$ action: the $\mathbb{C}^\times$ action on target $\mathbb{P}^1$ induces an action of $\mathbb{C}^\times$ on the moduli space by composition. $\lambda \in \mathbb{C}^\times$ then

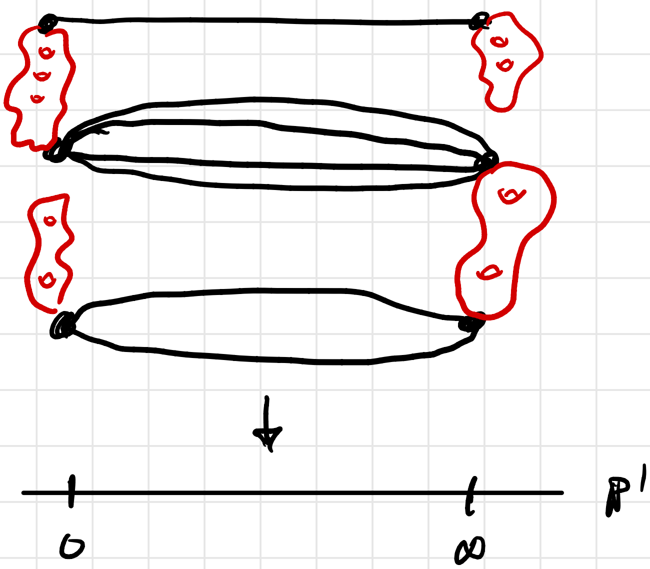
$$\lambda \cdot [f : C \to \mathbb{P}^1] = [C \xrightarrow{f} \mathbb{P}^1 \xrightarrow{\lambda} \mathbb{P}^1]$$

Integration on a smooth proj. manifold with a $\mathbb{C}^\times$ can be done by Atiyah-Bott localization. Integral can be computed purely by contributions from the $\mathbb{C}^\times$ fixed locus.

(pairing coh classes against the fundamental class)

There is a virtual version of Atiyah-Bott localization (Graber-Pandharipande) which reduces the above integral to an integral over the fixed locus. $\overline{M}_h(\mathbb{P}^1, d[\mathbb{P}^1])^{\mathbb{C}^\times}$

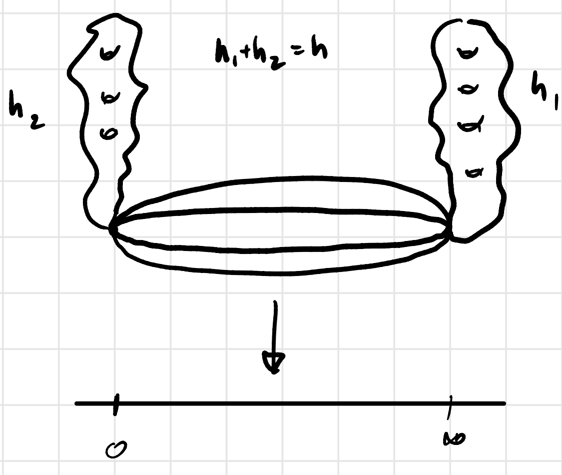
What kind of maps are $\mathbb{C}^\times$ fixed?



Domain has $\mathbb{P}^1$ components mapping to $\mathbb{P}^1$ with degree d_i, fully ramified over $0, \infty$
$\sum d_i = d$.

All other components collapse

Combinatorially complicated, but each fixed component is smooth and expressible in terms of $\overline{M}_{g_i, n_i}$'s. One can show the only fixed components that contribute are the simplest



$\cong \overline{M}_{h_1,1} \times \overline{M}_{h_2,1}$

this contributes $\frac{1}{d} \cdot \left(d^{2h_1 - 1} b_{h_1}\right)\left(d^{2h_2 - 1} b_{h_2}\right)$

where b_g are the Bernoulli numbers

$$\sum_{h \geq 0} b_h t^{2h} = \left(\frac{\sin(t/2)}{t/2}\right)^{-1}$$

The corresponding localization computation is easier in Donaldson-Thomas theory.

Potential and Partition function of local $\mathbb{P}^1$: $\quad X = \text{total}(\mathcal{O}(-1)\oplus\mathcal{O}(-1))$

$$N_{g,d} = N^{GW}_{g,d[\mathbb{P}^1]}(X)$$

$$F' = \sum_{\substack{g,d\\ d\neq 0}}' N_{g,d}\,\lambda^{2g-2} v^d = \sum_{\substack{g_1,g_2\geq 0\\ d>0}}' b_{g_1} b_{g_2}\, d^{2g_1+2g_2-3}\lambda^{2g_1+2g_2-2} v^d$$

prime means no degree 0 term

switch from q to v

$$= \sum_{d>0}' d^{-3}\lambda^{-2} v^d \sum_{g_1\geq 0} b_{g_1}(d\lambda)^{2g_1} \sum_{g_2\geq 0}' (d\lambda)^{2g_2}$$

$$= \sum_{d>0}' d^{-3}\lambda^{-2} v^d \left(\frac{\sin\left(\frac{d\lambda}{2}\right)}{d\lambda/2}\right)^{-2}$$

$$F'_X = \sum_{d>0}' \frac{v^d}{d}\left(2\sin\left(\frac{d\lambda}{2}\right)\right)^{-2} \leadsto F'_0 = \sum_{d=1}^{\infty}{}' \frac{v^d}{d^3}$$

Note that $\left(2\sin\left(\frac{d\lambda}{2}\right)\right)^2 = 4\sin^2\frac{d\lambda}{2} = 2(1-\cos d\lambda) = 2\left(1-\frac{1}{2}(e^{i\lambda d}+e^{-i\lambda d})\right)$

$$= 2 - e^{i\lambda d} - e^{-i\lambda d} \qquad \text{Let } q = e^{i\lambda}$$

$$= 2 - q^d - q^{-d} = -q^{d}(1-2q^d+q^{2d})$$

$$= -q^{-d}(1-q^d)^2$$

So $\displaystyle F'_X = \sum_{d=1}^{\infty}{}' \frac{-v^d}{d}\frac{q^d}{(1-q^d)^2}$ and since $\displaystyle \frac{x}{(1-x)^2} = x+2x^2+3x^3+\cdots$

$$\log(1-x) = \sum_{k=1}^{\infty} \frac{-x^k}{k}$$

$$= \sum_{d=1}^{\infty}{}' \sum_{k=1}^{\infty}{}' -k\frac{(q^k v)^d}{d}$$

$$= \sum_{k=1}^{\infty} k\log(1-q^k v) = \log\prod_{k=1}^{\infty}(1-vq^k)^k$$

So $\displaystyle Z'_X = \exp(F'_X) = \prod_{k=1}^{\infty}(1-vq^k)^k$

Example Super rigid elliptic curve if $E \subset X$ is an elliptic curve in a CY3 and

$N_{E/X} \cong L \oplus L^{-1}$ L generic degree 0 line bundle so $H^0(E, L^k) = 0$ for all $k \in \mathbb{Z}$.

Then $E \subset X$ is superrigid and the contribution of $E \subset X$ to the GW invs makes sense

Compute $N_{g,d}(\text{local } E)$.

$N_{g,d}(\text{local } E) = 0$ if $g \neq 1$ ⟵ Fact proven with degeneration (easier in DT theory)

Lecture 16

HW: Using covering space theory show there are $\sigma(d) = \sum_{k|d} k$ covering spaces of E.

$\Rightarrow$ $\overline{M}_1(E, d[E])$ consists of $\sigma(d)$ points each with automorphism group $\mathbb{Z}/d$.

$\Rightarrow$ $N_{1,d}(\text{local } E) = \frac{1}{d}\sigma(d)$

$$F' = \sum_{g=0}^{\infty}\sum_{d=1}^{\infty} N_{g,d}(\text{local } E)\, \lambda^{2g-2} v^d$$

$$= \sum_{d=1}^{\infty} v^d \frac{1}{d}\sigma(d) = \sum_{d>0}\sum_{k|d} \frac{k v^d}{d} \qquad d = k\cdot m$$

$$= \sum_{k,m>0} \frac{1}{m} v^{km}$$

$$= \sum_{k>0} -\log(1-v^k) =$$

$$= \log \prod_{k=1}^{\infty} \frac{1}{1-v^k}$$

$$\Rightarrow \quad Z'_{\text{local } E} = \prod_{k=1}^{\infty}\frac{1}{1-v^k} = \sum_{n=0}^{\infty} p(n) v^n \qquad p(n) = \#\text{ of partitions of } n.$$

$\Rightarrow$ $p(n) = \#$ of possibly disconnected (unramified) covers of degree n. Fun to prove with gp theory.

Proceeding along these lines one can imagine computing the multiple cover / degenerate map contributions for all local curves in order to obtain a "universal multiple cover formula" — something where there are integer counts of curves on X, say $n_{g,\beta}(X)$ and a universal formula (i.e. independent of X) relating $\{n_{g,\beta}(X)\}$ to $\{N_{g,\beta}(X)\}$.

A potential solution to this came early in the subject from physics.

In 1998 Gopakumar & Vafa defined <u>integer</u> curve counting invariants $n_{g,\beta}(X)$ (via counting BPS states in physics) and conjectured a formula relating them to GW invariants:

$$\boxed{F'_X = \sum_{g\geq 0}' \sum_{\beta\neq 0}' N^{GW}_{g,\beta}(X)\, \lambda^{2g-2} v^{\beta} = \sum_{g\geq 0}' \sum_{\beta\neq 0}' n_{g,\beta}(X) \sum_{k>0}' \frac{1}{k}\left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2} v^{k\beta}}$$

moreover, for fixed β, $n_{g,\beta}=0$ for $g>>0$.

<u>example</u> $$n_{g,d[\mathbb{P}^1]}(\text{local } \mathbb{P}^1) = \begin{cases} 1 & g=0, d=1 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ Gopakumar-Vafa invariants count each superrigid $\mathbb{P}^1$ once.

$$n_{g,d[E]}(\text{local } E) = \begin{cases} 1 & g=1, d \text{ any} \\ 0 & g\neq 1 \end{cases}$$

$\Rightarrow$ GV invariants count each superrigid Elliptic curve once in each class $d[E]$.

One can take the GV formula as the <u>definition</u> of $n_{g,\beta}(X)$ ($n_{g,\beta}$ is a linear combination of $N_{g',\beta'}$ for $g'\leq g$, $\beta'|\beta$). For years, this was how $n_{g,\beta}$ were defined — making the physics definition in to a mathematical geometric definition took 20 years (Maulik-Toda).

If we use the GV formula to define the GV invariants $n_{g,\beta}$, then there is no a priori reason to think they are integers. This was proved fairly recently:

<u>Theorem</u> Let the numbers $\{n_{g,\beta}(X)\}$ be defined in terms of the GW invariants $\{N_{g,\beta}(X)\}$ via the GV formula. Then

$$① \ n_{g,\beta}(X) \in \mathbb{Z} \quad \text{and} \quad ② \ n_{g,\beta}(X) = 0 \quad \text{for} \quad g > c(\beta).$$

(Parker-Ionel, Doan-Ionel-Walpuski 2021)

The proof of this uses symplectic / almost cx geometry to reduce the problem to local curves: $X = \mathrm{tot}(L \oplus L^{-1} \otimes K_C \to C)$ which can be computed in the algebraic category.

(B.-Pandharipande 2008)

Take aways: • even the integer invariants $n_{g,\beta}(X)$ are not just naive counts for $g>0$, a completely naive count (say in the symplectic category would not be deformation invariant).

• The geometric definition (Maulik and Toda) has the integers $n_{g,\beta}$ as dimensions of certain cohomology gps on a moduli space of sheaves. The GV formula then relates GW theory (stable maps / virtual classes) to a kind of DT theory (sheaves / cohomology).

Lecture 17,

The next step in understanding curves on CY3s is to use moduli spaces parameterizing sheaves (Donaldson-Thomas theory). Before we do that, lets do another spectacular application of GW theory to "classical" enumerative geometry

K3 surfaces and the Yau-Zaslow formula.

Recall a K3 surface is a CY surface that is simply connected (so not Abelian surface)

e.g. $X_{(4)} \subset \mathbb{P}^3$, $X_{(2,3)} \subset \mathbb{P}^4$, $X_{(2,2,2)} \subset \mathbb{P}^5$

Although all projective K3 surfaces are deformation equivalent and hence diffeomorphic, they come in families indexed by a number $n \in \mathbb{N}$ (a "genus").

<u>Def'n</u> A projective K3 surface X is of genus n if there exists a primitive curve class β with $\beta^2 = 2n-2$. Equivalently, there exists a map $X \to \mathbb{P}^n$ (embedding for $n \geq 3$) which does not factor through a smaller projective space.

Note that β is the hyperplane section and a generic hyperplane section will be a smooth curve of genus n.

There are a finite number of rational curves in the class β (i.e. hyperplane sections). For generic X, these rational curves will have n nodes. (Xi Chen)

r_n = # of rational hyperplane sections of a genus n K3 surface X.

In 1995, Yau-Zaslow conjectured the following amazing formula:

$$\sum_{n=0}^{\infty} r_n q^{n-1} = q^{-1} \prod_{n=1}^{\infty} (1-q^n)^{-24} = \Delta(q)^{-1}$$

$$= q^{-1} + 24 + 324q + 3200q^2 + \cdots$$

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \qquad q = e^{2\pi i\tau} \qquad \tau \in \mathbb{H} = \{\tau \in \mathbb{C} \ \ \mathrm{Im}\,\tau > 0\}$$

unique modular cusp form of weight 12. $\Delta(-1/\tau) = \tau^{12}\Delta(\tau)$

$\mathbb{H}/SL_2\mathbb{Z} \cong M_{1,1}$ $\quad \overline{M}_{1,1} = \mathbb{P}(4,6) = \mathrm{Proj}\, \mathbb{C}[E_4, E_6]$, Δ is the unique section of $\mathcal{O}(1) \to \overline{M}_{1,1}$ vanishing at $[\,\not{\!\!\infty}\,] \in \overline{M}_{1,1}$

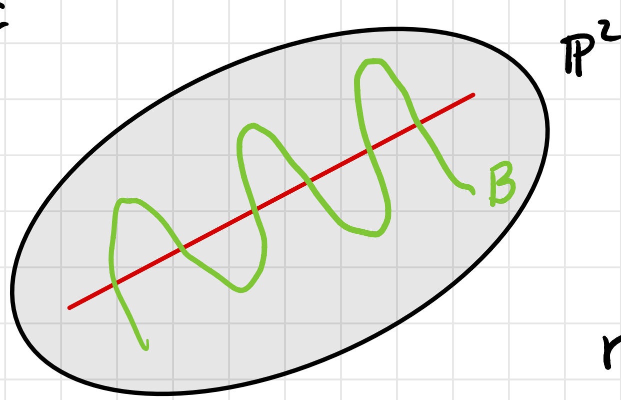
We will prove this with GW theory taking advantage of deformation invariance.

<u>examples</u>: $X_4 \subset \mathbb{P}^3$ a hyperplane section $H \cap X_4$ is a quartic curve in $H \cong \mathbb{P}^2$ and so is genus 3. If H is tangent to X_4 then $H \cap X_4$ has a nodal singularity so $r_3 = \#$ of tritangent planes to $X_4 \subset \mathbb{P}^3$ (3200)

Special Cases of $n=2$, $n=1$: A genus 2 K3 surface is a double branched cover $X \xrightarrow{\pi} \mathbb{P}^2$ branched over a smooth sextic curve B. A "hyperplane section" is then $\pi^{-1}(\text{line}) = C$

$C \xrightarrow{2:1} \mathbb{P}^1$ branched at 6 pts

hence C is genus 2

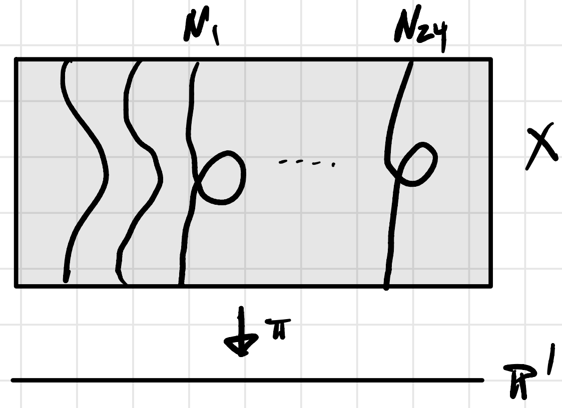


If the line is tangent to B, then cover will have a node so

$r_2 = \#$ of bitangent lines to a smooth sextic (324 by Plucker).

$n=1$ $\quad X \xrightarrow{\pi} \mathbb{P}^1$ is an elliptically fibered K3 surface. The "hyperplane section" is $\pi^{-1}(pt) = $ fiber (generically genus 1).

$r_1 = \#$ of rational fibers $= 24$



Lecture 18

<u>Problem</u>: The ordinary GW invariants of a K3 surface are zero. Why?

Given a K3 surface X and a class $\beta \in H^2(X;\mathbb{Z})$ which is algebraic (i.e. $\exists$ $C \subset X$ with $\beta = [C]$), there exists a deformation of X which makes β non-algebraic.

To be precise: a deformation is a 3-fold $\mathcal{X} \xrightarrow{\pi} \mathbb{A}^1$ whose fibers $\mathcal{X}_t = \pi^{-1}(t)$ are K3 surfaces and $\mathcal{X}_0 = X$. $H^2(\mathcal{X};\mathbb{Z}) \cong H^2(\mathcal{X}_t;\mathbb{Z})$ for all t

so given $\beta \in H^2(\mathcal{X}_0)$ it makes sense to talk about the same $\beta \in H^2(\mathcal{X}_t)$

The statement is that β is an algebraic class in $H^2(\mathcal{X}_0)$ but not in $H^2(\mathcal{X}_{t\neq 0})$

(algebraic classes are $H^2(X;\mathbb{Z}) \cap H^{1,1}(X;\mathbb{C})$ $\quad H^2(X;\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$)

So $\overline{M}_g(X,\beta) \neq \emptyset$ but $\overline{M}_g(\mathcal{X}_{t\neq 0},\beta) = \emptyset$ this implies any GW invariant of X in the class of β is 0 since by deformation invariance they are equal to the invariants of $\mathcal{X}_{t\neq 0}$.

Solution: Use the threefold $\mathcal{X}$ as our target! $\mathcal{X}$ is a CY3 and $\overline{M}_g(\mathcal{X},\beta) = \overline{M}_g(X,\beta)$ since any map to $\mathcal{X}$ in the class β must live in the central fiber. We may define:

$$r_\beta(X) = \int_{[\overline{M}_0(\mathcal{X},\beta)]^{vir}} 1 = \int_{[\overline{M}_0(X,\beta)]^{redvir}} 1$$

reduced virtual class. Invariant under deformations of X leaving β algebraic.

If X is a generic genus n K3 surface and β is the hyperplane class, then the above is an honest enumeration of the rational curves in the class β: since β is primitive there are no multiple covers. Since all rational curve are nodal (by Xi Chen for X generic) there are no collapsing components: every map is the normalization of its image.

Further analysis shows that $r_\beta(X)$ is invariant under deformations of X which leave β algebraic.

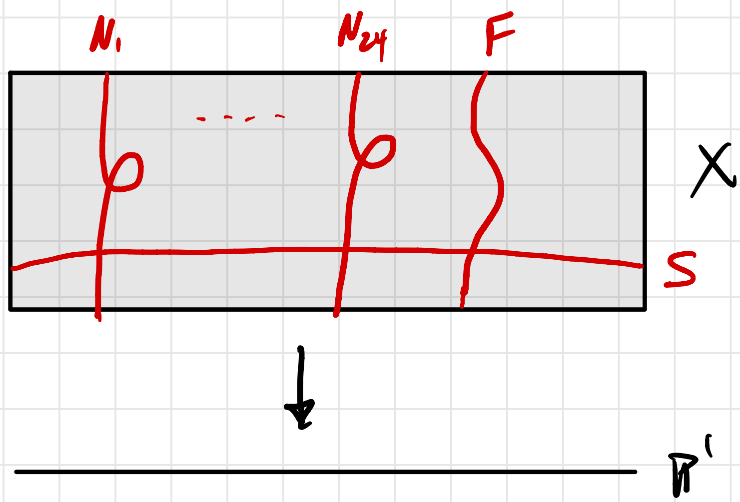
How do we compute? We now <u>weaponize deformation invariance</u>: we deform (X,β) from a generic genus n K3 surface to a very special one where we have a good handle on all the curves in the class β.

We know enough about the moduli space of K3 surfaces to know that a pair (X,β) consisting of a K3 surface and an effective curve class is deformation equivalent to any other (X',β') as long as β' has the same square and divisibility.

This means that ① $r_\beta(X)$ only depends on n where $\beta^2 = 2n-2$
"r_n"
(since β is primitive) and ② To compute r_n, we are free to choose any (X,β) with β primitive and $\beta^2 = 2n-2$.

Let X be an elliptically fibered K3 surface with a section and 24 nodal fibers. X can be constructed by taking $S_{(3,1)} \subset \mathbb{P}^2 \times \mathbb{P}^1$ generic rational elliptic surface and then pulling back by a generic 2:1 cover of the base

$$\begin{tikzcd} X \arrow[r, "2:1"] \arrow[d] & S \arrow[d] \\ \mathbb{P}^1 \arrow[r, "2:1"] & \mathbb{P}^1 \end{tikzcd}$$



Let $\beta_n = S + nF$ $\quad S^2 = -2 \quad F^2 = 0 \quad S \cdot F = 1$ so

$$\beta_n^2 = (S+nF)^2 = S^2 + 2n S\cdot F + F^2 = -2+2n \qquad \beta_n \text{ is primitive}$$

Lecture 19

This class is great because we can see all the curves in the associated linear system:

As we've said before, the linear system associated to an effective class of square $2n-2$ has dim n (follows from Hirzebruch-Riemann-Roch). This can be identified with $\mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$ where the divisor associated to $\{x_1, \dots, x_n\} \in \mathrm{Sym}^n \mathbb{P}^1$ is $S + \sum_{i=1}^{n} F_{x_i}$ where $F_{x_i} = \pi^{-1}(x_i)$.

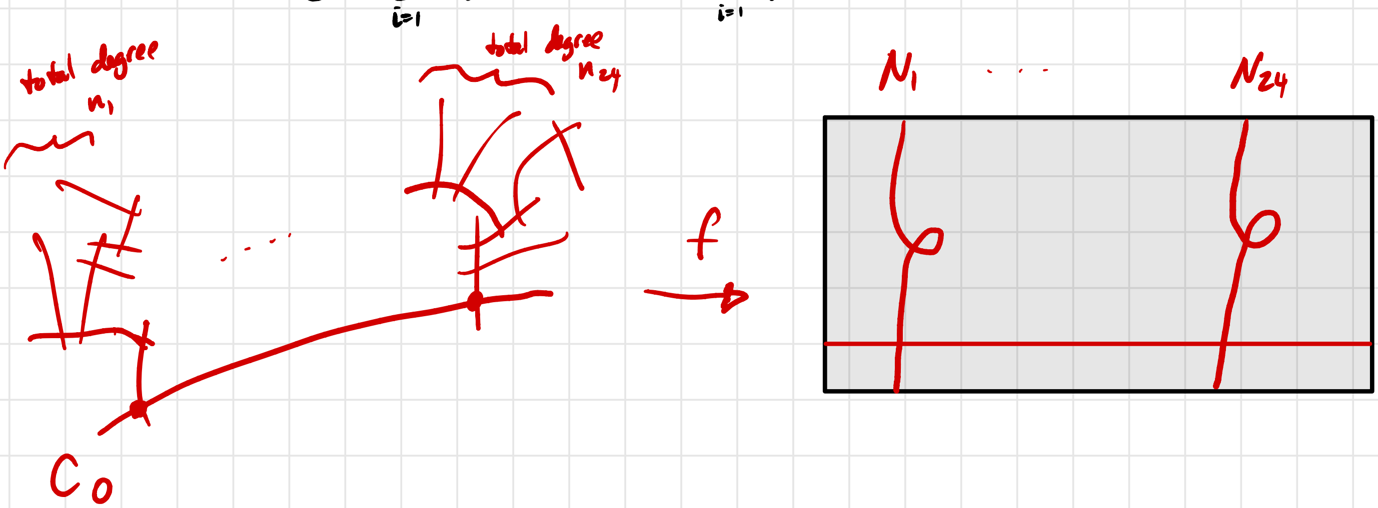
The price we pay for choosing this K3 and curve class is that we now must deal with multiple covers. What does $\overline{M}_0(X, \beta_n)$ look like?

Since the image of the map $f: C \to X$ is $S + \sum_{i=1}^{n} F_{x_i}$, we deduce that

① there is a component $C_0 \subset C$ such that $f|_{C_0}: C_0 \xrightarrow{\cong} S$

② Since C is genus 0, the dual graph is a tree and each subtree obtained by deleting the vertex corresponding to C_0 is a map of some tree of rational curves onto some fiber.

③ Since there are no non-constant maps from a rational curve to an elliptic curve, the image must be $S + \sum_{i=1}^{24} n_i N_i$ where $\sum_{i=1}^{24} n_i = n$.



$\Rightarrow$ moduli space is a product $\overline{M}_0(X, \beta_n) = \sum_{n_1 + \cdots + n_{24} = n} \prod_{i=1}^{24} \overline{M}_0(\mathcal{X}, S + n_i N_i)$

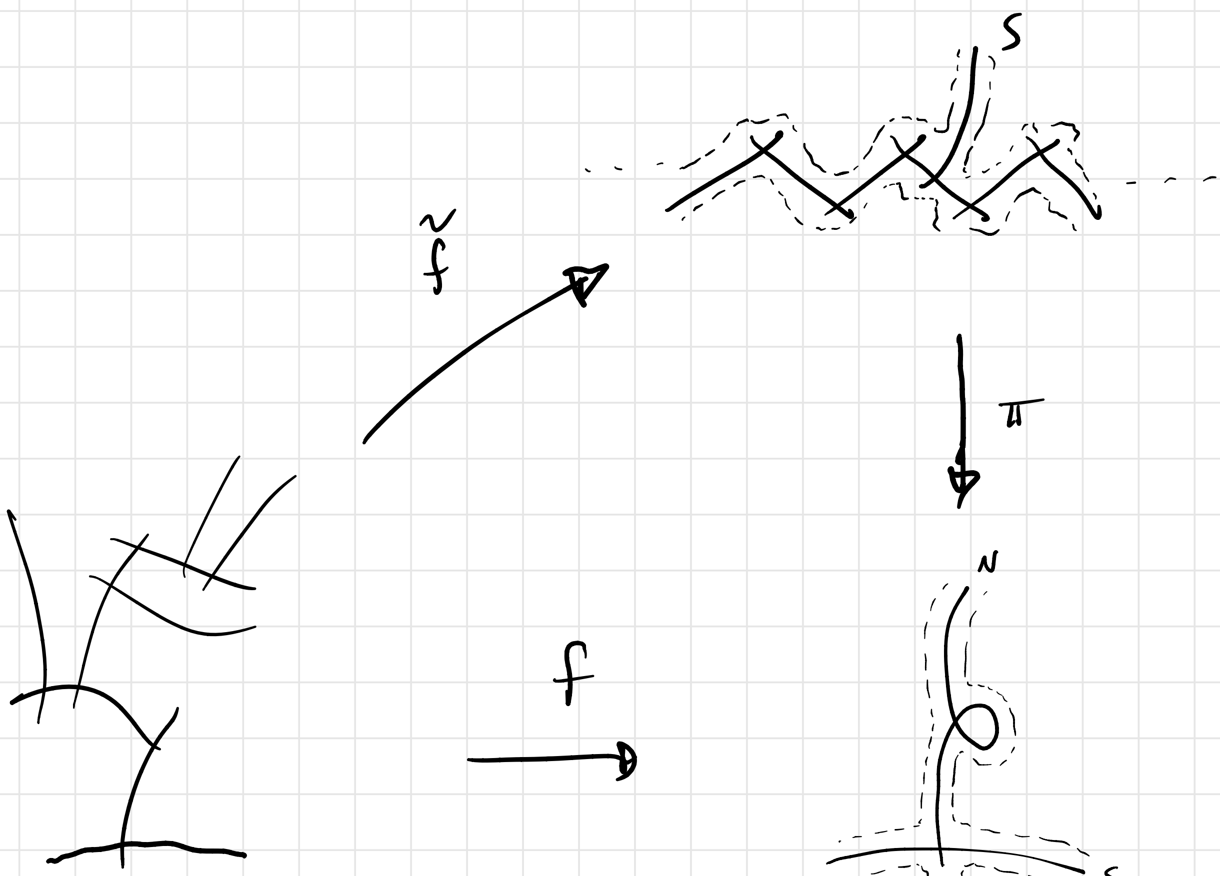
virtual class is also a product. Let $p(n) = \int_{[\overline{M}_0(\mathcal{X}^{N}, S + nN)]^{vir}} 1$

Then $r_n = \sum'_{n_1 + \cdots + n_{24} = n} \prod_{i=1}^{24} p(n_i)$

$$\Rightarrow \sum_{n=0}^{\infty} r_n q^n = \sum_{n_1,\dots,n_{24}} \prod_{i=1}^{24} q^{n_i} p(n_i) = \prod_{i=1}^{24} \left(\sum_{n_i=0}^{\infty} p(n_i) q^{n_i} \right)$$
$$= \left(\sum_{n=0}^{\infty} p(n) q^n \right)^{24}$$

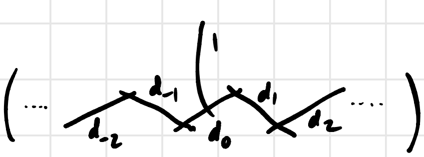
To prove Yau-Zaslow, we need to show $\sum p(n) q^n = \prod_{k=0}^{\infty} (1-q^k)^{-1}$ i.e.

$$p(n) = \int_{[\overline{M}_0(\ ,\, s+nN)]^{vir}} 1 = \#\text{ of partitions of } n.$$



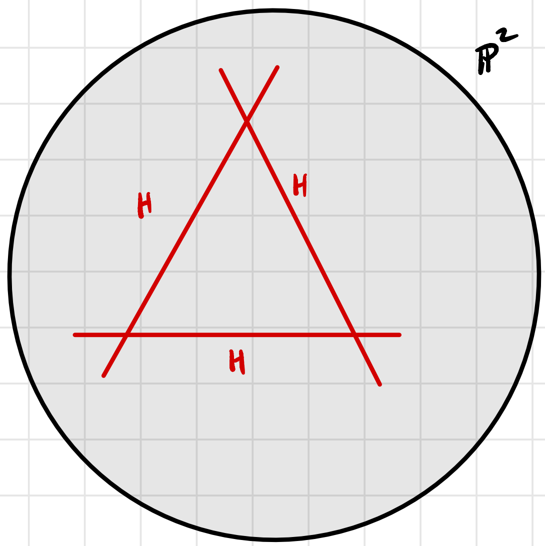
each stable map uniquely factors through the universal cover so

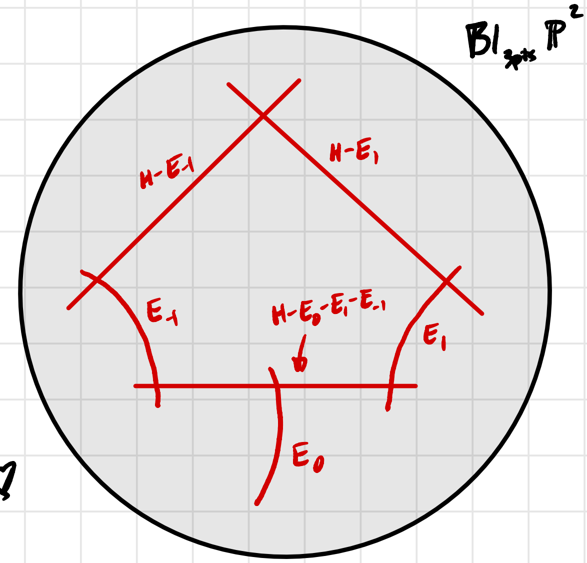
$$\overline{M}_0(\ ) = \bigsqcup_{\substack{\dots d_{-2}, d_{-1}, d_0, d_1, \dots \\ \sum d_i = n}} \overline{M}_0(\cdots\ )$$

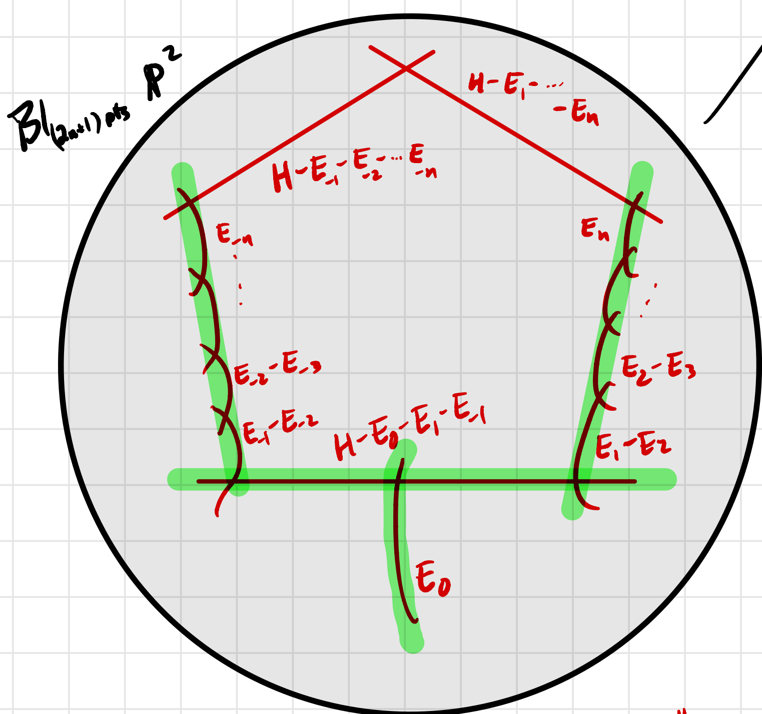


tree of rational curves in a surface, normal bundles are $\mathcal{O}(-2)$

Still hard to compute directly, but we can find the same configuration of curves in a very special blowup of $\mathbb{P}^2$:







$$\beta_{\vec{d}} = E_0 + d_0(H-E_0-E_1-E_{-1}) + \sum_{i\geq 1} d_i(E_i-E_{i+1}) + d_{-i}(E_{-i}-E_{-i-1})$$

all maps lie in configuration

$$\int_{[\overline{M}_0(\cdots\ d_{-2},\ d_{-1},\ d_0,\ 1,\ d_1,\ d_2\ \cdots)]^{vir}} 1 = \int_{[\overline{M}_0(Bl\,\mathbb{P}^2,\ \beta_{\vec{d}})]} 1$$

can be computed by Cremona, answer is always 0 or 1 !

Prop

$$\int_{[\overline{M}_0(Bl_-\mathbb{P}^2,\, \beta_{\vec{d}})} 1 = \begin{cases} 1 & \text{if } \cdots d_{-2} \le d_{-1} \le d_0 \ge d_1 \ge d_2 \cdots \text{ and } |d_i - d_{i-1}| = 0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases}$$

same or drops by 1

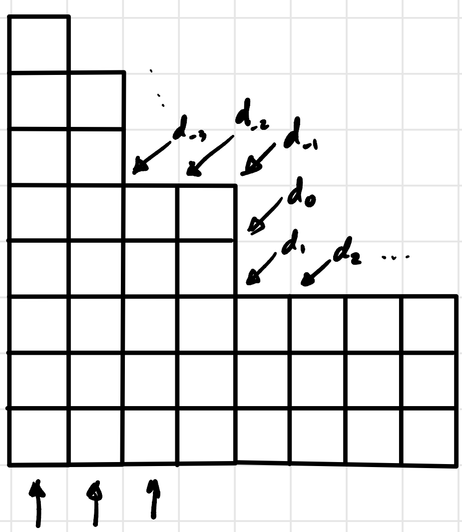
$$(\cdots\ d_{-2},\ d_{-1},\ d_0,\ d_1,\ d_2, \cdots)$$

biggest: d_0

Call such a sequence "admissible"

$$p(n) = \int_{[\overline{M}_0(\tilde{\mathcal{X}}^N,\, s + nN)]^{vir}} 1 = \#\text{ of admissible sequences } \vec{d} \text{ with } \sum_i d_i = n$$

Bijection between admissible sequences of size n and partitions of size n:



$n_1 \ge n_2 \ge n_3 \cdots \qquad \sum n_i = n$