Lecture 33 | DT/GW/GV invariants for local $\mathbb{P}^2$ $\quad \beta = [L]$ & $\beta = 2[L]$

$$X = \text{total}\left(\mathcal{O}(-3) \longrightarrow \mathbb{P}^2\right)$$

$$Z^{DT}_\beta(X) = \sum_n{}' N^{DT}_{\beta,n}(X)\,(-q)^n$$

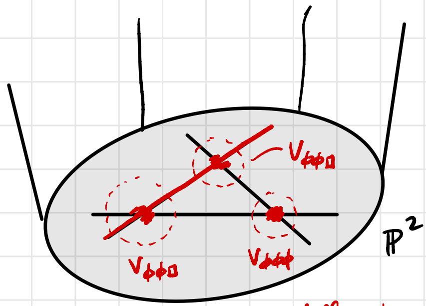
$$= \sum_n e\left(I_n(X,\beta),\nu\right)(-q)^n$$

value of Behrend fnc on torus fixed points $[I_Z] \in I_n(X,\beta)^T$ is $(-1)^{c(\beta)+n}$

$$= \sum_n (-1)^{c(\beta)+n}\,\#\{I_n(X,\beta)^T\}\,(-q)^n$$

$$= \pm \sum_n{}' \#\{I_n(X,\beta)^T\}\,q^n$$

$$Z^{DT}_{[L]} = \pm\, 3\, V^2_{\phi\phi\square}(q)\cdot V_{\phi\phi\phi}(q)\cdot q^{\chi(\mathcal{O}_L)}$$



3 different places for line

$$V_{\phi\phi\phi} = M(q) = \prod_{m=1}^{\infty}(1-q^m)^{-m}$$

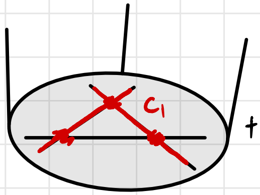
$$V_{\phi\phi\square} = M(q)\,\frac{1}{1-q}$$

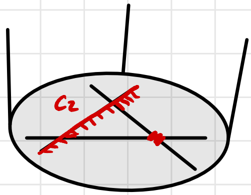
$$Z_{[L]} = \pm 3\, M(q)^3\,\frac{1}{(1-q)^2}\,q^1 = \pm 3q + O(q^2)$$

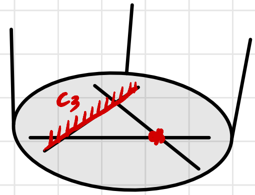
$I_1(X,[L]) = \mathbb{P}^2$ so $N_{1,[L]} = e_{vir}(\mathbb{P}^2) = 3$ so

$$Z_{[L]} = -3\,M(q)^3\,\frac{q}{(1-q)^2}$$

Recall $Z'_\beta = \dfrac{Z_\beta}{Z_0}$ so $\boxed{Z'_{[L]} = \dfrac{-3q}{(1-q)^2}}$ Invariant under $q \leftrightarrow q^{-1}$

$$Z_{2[L]} = \pm \Big\{ 3$$ 

$$+\ 3$$ 

$$+\ 3$$ 

$$\Big\}$$

$$= \pm \left(3 V_{\phi\phi\square}^2 V_{\phi\square\square}\, q^{\chi(\mathcal{O}_{C_1})+1} + 3 V_{\phi\phi\square}^2 V_{\phi\phi\phi} \left(q^{\chi(\mathcal{O}_{C_2})} + q^{\chi(\mathcal{O}_{C_3})} \right) \right)$$

$$V_{\phi\phi\square} = V_{\phi\phi\phi} \frac{1}{1-q} \qquad V_{\phi\square\square} = V_{\phi\phi\phi}\left(q^{-1} + \frac{1}{(1-q)^2} \right) \qquad V_{\phi\phi\boxminus} = V_{\phi\phi\phi} \frac{1}{(1-q)(1-q^2)}$$

$\chi(\mathcal{O}_{C_1}) = \chi(\mathcal{O}_{C_2}) = 1$ since $C_1, C_2 \subset \mathbb{P}^2 \subset X$ conics so have $\chi = 1$ by adjunction

we will see how to compute $\chi(\mathcal{O}_{C_3})$ later $\chi(\mathcal{O}_{C_3}) = 5$

$$Z_{2[L]} = \pm 3 M(q)^3 \left(\frac{1}{(1-q)^2} \left(q^{-1} + \frac{1}{(1-q)^2} \right) q^2 + \frac{q + q^5}{(1-q)^2 (1-q^2)^2} \right)$$

$$Z'_{2[L]} = \pm 3 \frac{q}{(1-q)^2} \left[1 + \frac{q}{(1-q)^2} + \frac{1+q^4}{(1-q^2)^2} \right]$$

Invariant under $q \leftrightarrow q^{-1}$? yes!

$$Z_{2[L]} = \pm 3 \left(q + O(q^2) \right) \left(1 + q + O(q^2) + 1 + O(q^2) \right) M(q)^3 = \pm 6 \left(q + O(q^2) \right)$$

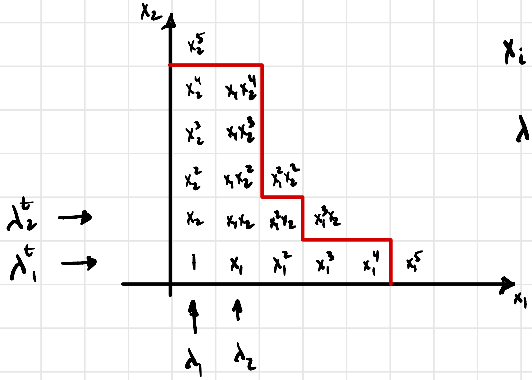
so $\pm 6 = -N_{1,2[L]} = -e_{vir}(I_1(X, 2[L])) = -e_{vir}(\mathbb{P}^5) = -(-6) = +6$

$$\boxed{Z'_{2[L]} = 3 \frac{q}{(1-q)^2} \left[1 + \frac{q}{(1-q)^2} + \frac{1+q^4}{(1-q^2)^2} \right]}$$

Digression on how to compute $\chi(\mathcal{O}_{C_\lambda})$:

Let $X = \mathrm{tot}\left(\mathcal{O}_{\mathbb{P}^1}(-m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m_2)\right)$ $\quad m_1 + m_2 = 2 \quad$ (in our case $m_1 = 1$, $m_2 = -3$)

Let C_λ be the pure dim'l 1 subscheme of X (no embedded points) which is T invariant and restricted to the fibers of $X \xrightarrow{\pi} \mathbb{P}^1$ is the monomial subscheme of $\mathbb{C}^2$ given by λ.



x_i is coordinate on the fibers of $\mathcal{O}(-m_i)$

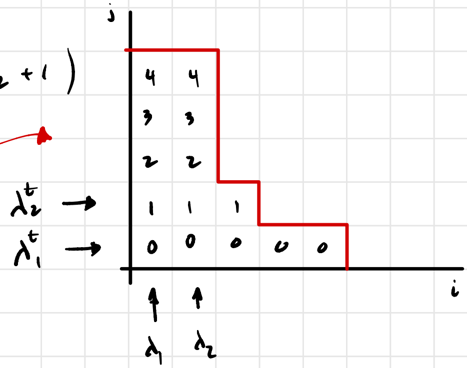
$\lambda = (5,5,2,1,1)$ $\quad \lambda^t = (5,3,2,2,2)$

C_λ restricted to fiber is cut out by $(x_2^5, x_1^2x_2^2, x_1^3x_2, x_1^5)$

To compute $\chi(\mathcal{O}_{C_\lambda})$ we compute $\chi(\pi_*\mathcal{O}_{C_\lambda})$ [In general, for $f: V \to W$ $\mathcal{G}$ a sheaf on V, $\sum_k (-1)^k \dim H^k(V, \mathcal{F}) = \sum_{i,j}' (-1)^{i+j} \dim H^i(W, R^j f_* \mathcal{G})$, since $R^j\pi_*\mathcal{O}_{C_\lambda} = 0$ for $j > 0$, $\chi(\mathcal{O}_{C_\lambda}) = \chi(\pi_*\mathcal{O}_{C_\lambda})$].

$\pi_*\mathcal{O}_{C_\lambda}$ is a bundle of rank $|\lambda|$ on $\mathbb{P}^1$. If E is a vector space then of course elts of $E^\vee$ are linear functions on E and hence $\mathrm{Sym}^n E^\vee$ are homogeneous degree n polynomial functions. If $E \xrightarrow{\pi} Y$ is a vector bundle then sections of $\mathrm{Sym}^n E^\vee$ are polynomial functions on the fibers of π. So for $E = \mathcal{O}(-m_1) \oplus \mathcal{O}(-m_2)$, the monomial $x_1^i x_2^j$ is a section of $\mathcal{O}(m_1)^{\otimes i} \otimes \mathcal{O}(m_2)^{\otimes j} = \mathcal{O}_{\mathbb{P}^1}(im_1 + jm_2)$

Thus $\pi_* \mathcal{O}_{C_\lambda} = \bigoplus_{i,j \in \lambda} \mathcal{O}(im_1 + jm_2)$



$$\Rightarrow \quad \chi(\mathcal{O}_{C_\lambda}) = \chi(\pi_* \mathcal{O}_{C_\lambda}) = \sum_{i,j \in \lambda}' (im_1 + im_2 + 1)$$

$$= |\lambda| + m_1 \sum_{i,j} i + m_2 \underbrace{\sum_{i,j \in \lambda} j}$$

$$= |\lambda| + m_1 \binom{\lambda^t}{2} + m_2 \binom{\lambda}{2}$$

In our case $m_1 = -1$ $m_2 = 3$ $\lambda = (2)$ $\lambda' = (1,1)$
or $\lambda = (1,1)$ $\lambda' = (2)$

$$\text{so } \chi(\mathcal{O}_{C_\beta}) = \begin{cases} 2 - 1 + 0 \\ 2 + 0 + 3 \end{cases} = 1 \text{ or } 5$$

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$$\boxed{Z'_{DT}(x) = 1 - \frac{3q}{(1-q)^2} v + \frac{3q}{(1-q)^2}\left[1 + \frac{q}{(1-q)^2} + \frac{1+q^4}{(1-q^2)^2}\right] v^2 + \mathcal{O}(v^3)}$$

We are done with geometry, we now can use DT/GW and GV formula to determine GW and GV invariants. Non trivial formula mongering.

$$Z'_{DT} \overset{q = e^{i\lambda}}{=} \exp(F'_{GW}) \overset{\text{GV formula}}{=} \exp\left\{ \sum_{\substack{g \geq 0 \\ \beta \neq 0 \\ k > 0}}' n_{g,\beta} \frac{v^{k\beta}}{k} \left(2 \sin \frac{k\lambda}{2}\right)^{2g-2} \right\}$$

$$\boxed{Z'_{DT} = \exp\left\{ \sum_{\substack{g \geq 0 \\ \beta \neq 0 \\ k > 0}}' n_{g,\beta} \frac{v^{k\beta}}{k} (-1)^{g-1} \left(\frac{q^k}{(1-q^k)^2}\right)^{1-g} \right\}}$$

We want $n_{g,d[L]}(X)$ for $d=1$ & 2 $\quad X = \text{total}(\mathcal{O}(-3)\to\mathbb{P}^2)$

$$Z'_{DT}(X) = 1 - \frac{3q}{(1-q)^2}\,v + \frac{3q}{(1-q)^2}\left[1+\frac{q}{(1-q)^2}+\frac{1+q^4}{(1-q^2)^2}\right]v^2 + O(v^3)$$

$$Z'_{DT}(X) = \exp\left\{\sideset{}{'}\sum_{\substack{g\geq 0\\ \beta\neq 0\\ k>0}} n_{g,\beta}\,\frac{v^{k\beta}}{k}\,(-1)^{g-1}\left(\frac{q^k}{(1-q^k)^2}\right)^{1-g}\right\}$$

$$Z'_{DT}(x) = \exp\left(c_1 v + c_2 v^2 + \cdots\right) = 1 + c_1 v + \left(c_2 + \tfrac{1}{2}c_1^2\right)v^2 + \cdots$$

$$c_1 = \sum_{g\geq 0} n_{g,[L]}\,(-1)^{g-1}\left(\frac{q}{(1-q)^2}\right)^{1-g}$$

$$c_2 = \sum_{g\geq 0} n_{g,[L]}\,(-1)^{g-1}\,\frac{1}{2}\,\overset{k=2}{\left[\frac{q^2}{(1-q^2)^2}\right]^{1-g}} + \sum_{g\geq 0} n_{g,2[L]}\,(-1)^{g-1}\overset{k=1}{\left(\frac{q}{(1-q)^2}\right)^{1-g}}$$

v^1 term: $\quad \dfrac{-3q}{(1-q)^2} = \displaystyle\sideset{}{'}\sum_{g=0}^{\infty} n_{g,[L]}\,(-1)^{g}\left(\frac{q}{(1-q)^2}\right)^{1-g}$

$$\Rightarrow\quad n_{g,[L]} = \begin{cases} 3 & g=0 \\ 0 & g>0 \end{cases}$$

v^2 term:

$$\frac{3q}{(1-q)^2}\left(1+\frac{q}{(1-q)^2}+\frac{1+q^4}{(1-q^2)^2}\right) = \frac{1}{2}c_1^2 + c_2 = \overset{\frac{1}{2}c_1^2}{\frac{9}{2}\,\frac{q^2}{(1-q)^4}} + \overbrace{-\frac{3}{2}\,\frac{q^2}{(1-q^2)^2} + \sum_{g} n_{g,2[L]}\,(-1)^{g-1}\left(\frac{q}{(1-q)^2}\right)^{1-g}}^{c_2}$$

$$\sum_{g} n_{g,2[L]}\,(-1)^{g-1}\left(\frac{q}{(1-q)^2}\right)^{1-g} = \frac{3q}{(1-q)^2}\cdot\left\{1 + \underbrace{\frac{q}{(1-q)^2}+\frac{1+q^4}{(1-q^2)^2} - \frac{3}{2}\,\frac{q}{(1-q)^2} + \frac{1}{2}\,\frac{q}{(1+q)^2}}_{\text{all the poles must cancel!}}\right\}$$

$$1+\frac{q}{(1-q)^2}+\frac{1+q^4}{(1-q^2)^2}-\frac{3}{2}\frac{q}{(1-q)^2}+\frac{1}{2}\frac{q}{(1+q)^2} = 1+\frac{1}{(1-q^2)^2}\left[q(1+q)^2+1+q^4-\frac{3}{2}q(1+q)^2+\frac{1}{2}q(1-q)^2\right]$$

$$= 1+\frac{1}{(1-q^2)^2}\left[\frac{1}{2}q(1-q)^2-\frac{1}{2}q(1+q)^2+1+q^4\right] = 1+\frac{1}{(1-q^2)^2}\left[-2q^2+1+q^4\right] = 1+1=2$$

$$\Rightarrow \sum_{g\geq 0}' n_{g,2[L]}\left(\frac{-q}{(1-q)^2}\right)^{1-g} = \frac{6q}{(1-q)^2}$$

$$\Rightarrow \quad n_{g,2[L]}(X) = \begin{cases} -6 & \text{if } g=0 \\ 0 & \text{if } g>0 \end{cases}$$

In our topological vertex computations, we exploited the fact that if M is a variety with the action of a torus, then $e(M)=e(M^T)$. We also want to use the fact that Euler characteristic is <u>motivic</u>: it behaves well under products and stratifications. There is a very nice way of formalizing this use the Grothendieck group:

<u>Def'n</u> The Grothendieck group of varieties over $\mathbb{C}$ is

$K_0(Var_{\mathbb{C}})$ = free Abelian group generated by isomorphism classes of varieties with the relation

$$[V]=[V-Z]+[Z] \quad \text{if } Z\subset V \text{ closed}$$

it is a ring under $[V]\cdot[W]=[V\times W]$. It has unit $1=[pt]$.

Remark: If
$$\begin{array}{c} F \to P \\ \downarrow \\ B \end{array}$$
is a Zariski locally trivial fibration, then $[P] = [F]\cdot[B]$ in $K_0(Var_{\mathbb{C}})$

pf Let $U = B - Z$ be a Zariski open set where P is trivial, then

$[P] = [P|_U] + [P|_Z] = [F \times U] + [P|_Z]$ by induction on the dimension of the base, we may assume $[P|_Z] = [Z]\cdot[F]$ and so

$$[P] = [F]\cdot[U] + [F]\cdot[Z] = [F]\cdot([U]+[Z]) = [F]\cdot[B]$$

Example $\mathbb{C}^\times \to \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ is Zar. locally trivial so

$[\mathbb{C}^{n+1} - \{0\}] = [\mathbb{C}^\times]\cdot[\mathbb{P}^n]$ Let $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$ $1 = [pt]$

$$\mathbb{L}^{n+1} - 1 = (\mathbb{L}-1)\cdot[\mathbb{P}^n] \Rightarrow [\mathbb{P}^n] = \frac{\mathbb{L}^{n+1}-1}{\mathbb{L}-1} = 1 + \cdots + \mathbb{L}^n$$

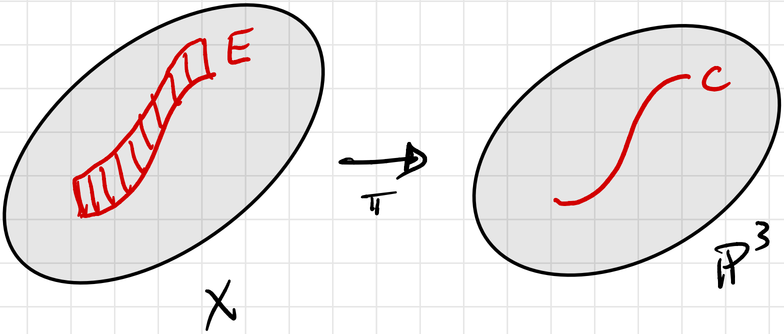
Prop: Euler characteristic is a ring homomorphism
$$e : K_0(Var_{\mathbb{C}}) \longrightarrow \mathbb{Z}$$
$$[V] \longmapsto e(V)$$

e.g. $e(\mathbb{P}^n) = n+1$

Theorem There exists a ring homomorphism $W_t : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}[t]$ (the weight polynomial) such that if V is a smooth projective variety then $W_t([V]) = \sum_i \dim H^i(V)\, t^i$ (the Poincare Poly). Moreover $W_t(\mathbb{L}) = t^2$.

e.g. $W_t(\mathbb{P}^n) = 1 + t^2 + \cdots + t^{2n}$

e.g. $X = Bl_C(\mathbb{P}^3)$ $\quad C \subset \mathbb{P}^3$ a smooth curve of genus g.



$E = \mathbb{P}(N_{C/\mathbb{P}^3}) \longrightarrow C$ with fibers $\mathbb{P}^1$ so $[E] = [\mathbb{P}^1]\cdot[C]$

$$[X] = [\mathbb{P}^3 - C] + [E]$$

$$= 1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3 - [C] + (1+\mathbb{L})[C] = 1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3 + \mathbb{L}\cdot[C]$$

so $W_t(X) = 1 + t^2 + t^4 + t^6 + t^2(1 + 2g\,t + t^2) = \underset{H^0}{1} + \underset{H^2}{2t^2} + \underset{H^3}{2g\,t^3} + \underset{H^4}{2t^4} + \underset{H^6}{t^6}$

Lecture 34

example: $[Gl_n(\mathbb{C})] = (\mathbb{L}^n - 1)\cdot(\mathbb{L}^n - \mathbb{L})\cdot(\mathbb{L}^n - \mathbb{L}^2)\cdots(\mathbb{L}^n - \mathbb{L}^{n-1})$

why? Gl_n has a fibration $\quad [Gl_n] = [\mathbb{L}^n - 1][F^1]$

$$\begin{array}{ccc} F^1_{v_1} & \longrightarrow & Gl_n \\ & & \downarrow \text{map matrix to first column} \\ v_1 & \in & \mathbb{C}^n - \{0\} \end{array}$$

Then $F^1_{v_1}$ has a fibration $\quad F^1 = [\mathbb{L}^n - \mathbb{L}^1]\cdot[F^2]$

$$\begin{array}{ccc} F^2_{v_2} & \longrightarrow & F^1_{v_1} \\ \downarrow & & \downarrow \text{2nd column} \\ v_2 & \in & \mathbb{C}^n - \{\text{span } v_1\} \cong \mathbb{C}^n - \mathbb{C} \end{array}$$

and so on:

$$\begin{array}{ccc} F^3_{v_3} & \longrightarrow & F^2_{v_2} \\ & & \downarrow \text{3rd column} \\ & & \mathbb{C}^n - \{\text{span } v_1, v_2\} \cong \mathbb{C}^n - \mathbb{C}^2 \end{array}$$

$$[GL_n(\mathbb{C})] = (\mathbb{L}^n-1)\cdot(\mathbb{L}^n-\mathbb{L})\cdot(\mathbb{L}^n-\mathbb{L}^2)\cdots(\mathbb{L}^n-\mathbb{L}^{n-1})$$

$$= \mathbb{L}^{\binom{n}{2}}\underbrace{(\mathbb{L}^n-1)(\mathbb{L}^{n-1}-1)\cdots(\mathbb{L}-1)}_{[n!]_{\mathbb{L}}} = \mathbb{L}^{\binom{n}{2}}[n!]_{\mathbb{L}}$$

$$Gr(k,n) = \frac{GL_n(\mathbb{C})}{\begin{array}{c} k\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\ \qquad\underbrace{}_{n-k} \end{array} = P}$$

$$\begin{array}{c} P \to GL_n(\mathbb{C}) \\ \downarrow \\ Gr(k,n) \end{array}$$

$$[P] = [GL_k]\cdot[GL_{n-k}]\cdot\mathbb{L}^{k(n-k)}$$

$$[Gr(k,n)] = \frac{\mathbb{L}^{\binom{n}{2}}[n!]_{\mathbb{L}}}{\mathbb{L}^{\binom{k}{2}}[k!]_{\mathbb{L}}\,\mathbb{L}^{\binom{n-k}{2}}[(n-k)!]_{\mathbb{L}}\cdot\mathbb{L}^{k(n-k)}}$$

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k) \text{ so}$$

$$[Gr(k,n)] = \frac{[n!]_{\mathbb{L}}}{[k!]_{\mathbb{L}}\,[(n-k)!]_{\mathbb{L}}} = \binom{n}{k}_{\mathbb{L}} = \frac{(\mathbb{L}^n-1)\cdots(\mathbb{L}^{k+1}-1)}{(\mathbb{L}^{n-k}-1)\cdots(\mathbb{L}-1)}$$

$$W_t(Gr(k,n)) = \frac{(t^{2n}-1)\cdots(t^{2k+2}-1)}{(t^{2n-2k}-1)\cdots(t^2-1)}$$

$$e(Gr(k,n)) = W_{-1}(Gr(k,n)) = \binom{n}{k}$$

$$Gr(2,5) = \frac{(\mathbb{L}^5-1)(\mathbb{L}^4-1)\cancel{(\mathbb{L}^3-1)}}{\cancel{(\mathbb{L}^3-1)}(\mathbb{L}^2-1)(\mathbb{L}-1)} = (\mathbb{L}^2+1)(\mathbb{L}^4+\mathbb{L}^3+\mathbb{L}^2+\mathbb{L}+1) \text{ so}$$

$$W_t(Gr(2,5)) = W_t(1+\mathbb{L}+2\mathbb{L}^2+2\mathbb{L}^3+2\mathbb{L}^4+\mathbb{L}^5+\mathbb{L}^6)$$

$$= \underset{H^0}{1} + \underset{H^2}{t^2} + \underset{H^4}{2t^4} + \underset{H^6}{2t^6} + \underset{H^8}{2t^8} + \underset{H^{10}}{t^{10}} + \underset{H^{12}}{t^{12}}$$