$K_0(\mathrm{Var}_\mathbb{C})$ has an even deeper structure: it is a lambda ring with $\sigma_k(X) = [\mathrm{Sym}^k X]$

One equivalent way to talk about (pre-) lambda ring structures is with <u>power structures</u>:

<u>**Def'n**</u> A <u>power structure</u> on a ring R is the following: for $A(t) \in R[[t]]$ with $A(0)=1$ and $M \in R$ $\exists$ $A(t)^M \in R[[t]]$ satisfying:

1. $A(t)^0 = 1$
2. $A(t)^1 = A(t)$
3. $A(t)^{M+N} = A(t)^M A(t)^N$
4. $A(t)^{M \cdot N} = (A(t)^M)^N$
5. $A(t)^M \cdot B(t)^M = (A(t) \cdot B(t))^M$
6. $(1+t)^M = 1 + Mt + O(t^2)$
7. $(A(t^k))^M = (A(t)^M)_{t \mapsto t^k}$

<u>Lecture 36</u>

<u>**Theorem**</u> (Getzler, Gusein-Zade et. al.) There exists a power structure on $K_0(\mathrm{Var}_\mathbb{C})$ uniquely determined by $(1-t)^{-[X]} = (1+t+t^2+\cdots)^{[X]} = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n$ if X is a <u>variety</u>.

<u>**Lemma**</u> (Totaro) $[\mathrm{Sym}^n \mathbb{C}^k] = [\mathbb{C}^{nk}]$ in $K_0(\mathrm{Var}_\mathbb{C})$. In terms of power structures

$$(1-t)^{-\mathbb{L}^k} = (1-\mathbb{L}^k t)^{-1}$$

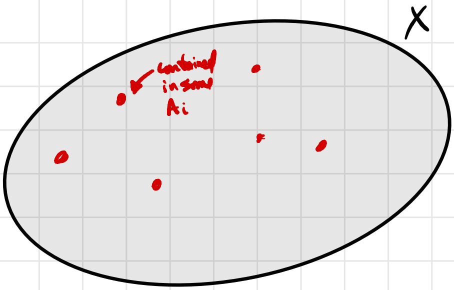
This theorem has a more general geometric interpretation: If $A(t) = \sum A_i t^i$ with A_i varieties, and X is a variety, then $(A(t))^X = \sum_k B_k t^k$ where

the index i is called "charge"

B_k = configurations of A_i-valued points in X of total charge k

$$= \left\{ (S, \phi) : S \subset X \text{ finite } \quad \phi : S \to \coprod_i A_i \quad \text{s.t. } k = \sum_{x \in S} i(\phi(x)) \right\}$$

$=$



particles on X of total charge k where the internal state space of a charge i particle is A_i.

$$= \coprod_{\alpha \vdash k} \left(\prod_{i=1}^{\infty} X^{b_i(\alpha)} \setminus \Delta \right) \times_{S_\alpha} \left(\prod_{i=1}^{\infty} A_i^{b_i(\alpha)} \right)$$

$b_i(\alpha) :=$ # of parts of size i $\qquad S_\alpha := \prod_{i=1}^{\infty} S_{b_i(\alpha)}$ ← symmetric groups

The theorem has no geometry: it is essentially set theoretic to see that there is a unique way to extend def'n to A_i and X not varieties such that ① - ⑦ hold.

Example: Let X be a smooth variety of dimension d, Let $\mathrm{Hilb}_0^i(\mathbb{A}^d_{\mathbb{C}}) \subset \mathrm{Hilb}^i(\mathbb{A}^d_{\mathbb{C}})$ be the locus of subschemes $Z \subset \mathbb{A}^d$ of length i supported at 0.

Then $\displaystyle \left(\sum_{i=0}^{\infty} [\mathrm{Hilb}_0^i(\mathbb{A}^d_{\mathbb{C}})] t^i \right)^{[X]} = \sum_{k=0}^{\infty} [\mathrm{Hilb}^n(X)] t^n$

Power structure is compatible with euler char homomorphism:

$e : K_0(Var_{\mathbb{C}})[[t]] \longrightarrow \mathbb{Z}[[t]]$ in the obvious way, then

$e(A(t)^X) = e(A(t))^{e(X)}$ so for example since

$e(\mathrm{Hilb}^i_0(\mathbb{A}^2)) = e(\mathrm{Hilb}^i_0(\mathbb{A}^2)^T) = p(i)$ # of partitions of i

$$\sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X))t^n = \left(\sum_{n=0}^{\infty} e(\mathrm{Hilb}^n_0(\mathbb{A}^2))\, t^n\right)^{e(X)} = \left(\sum_{n=0}^{\infty} p(n)t^n\right)^{e(X)}$$

$$= \prod_{m=1}^{\infty}(1-t^m)^{-e(X)}$$

Similarly, if X is a smooth 3-fold

$$\sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X))\, t^n = \left(\sum_{n=0}^{\infty} e(\mathrm{Hilb}^n_0(\mathbb{A}^3))t^n\right)^{e(X)} = \left(\sum_{n=0}^{\infty} P_{3D}(n)t^n\right)^{e(X)}$$

$$= M(t)^{e(X)} = \prod_{m=1}^{\infty}(1-q^m)^{-m e(X)}$$

$\Rightarrow Z_0^{DT}(X) = M(q)^{e(X)}$ (needs a little work to deal with Behrend fnc)

The power structure on $K_0(Var_{\mathbb{C}})$ is also compatible with the weight polynomial homomorphism $W_s : K_0(Var_{\mathbb{C}}) \longrightarrow \mathbb{Z}[s]$

where the power structure on $\mathbb{Z}[s]$ satisfies

$$(1-t)^{-(-s)^k} = (1-(-s^k)t)^{-1} \Rightarrow (1-t^m)^{-(-s)^k} = (1-(-s)^k t^m)^{-1}$$

We can use this to compute the betti numbers of $\mathrm{Hilb}^n(S)$ S a smooth surface (famous formula of Göttsche).

We start with getting the class of $\mathrm{Hilb}^n(\mathbb{C}^2)$ in $K_0(\mathrm{Var}_{\mathbb{C}})$. Let T be the torus and let $\mathbb{C}^\times \subset T$ be some generic subtorus. Then

$$\mathrm{Hilb}^n(\mathbb{C}^2)^{\mathbb{C}^\times} = \mathrm{Hilb}^n(\mathbb{C}^2)^T = p(n) \text{ points given by } Z_\lambda \subset \mathbb{C}^2 \quad \lambda \vdash n$$
defined by monomial ideal I_λ.

for any point $p \in \mathrm{Hilb}^n(\mathbb{C}^2)$ we can consider $\lim\limits_{\substack{t \to 0 \\ t \in \mathbb{C}^\times}} t \cdot p \in \mathrm{Hilb}^n(\mathbb{C}^2)^T$

Lecture 37

This defines a stratification $\mathrm{Hilb}^n(\mathbb{C}^2) = \bigsqcup\limits_{\alpha \vdash n} V_\alpha \leftarrow$ set of points limiting to $[Z_\alpha]$

It turns out that $V_\alpha \cong \mathbb{A}^{d(\alpha)}$ and $d(\alpha) = \#$ positive weight $\mathbb{C}^\times$ reprs in $T_{[Z_\alpha]} \mathrm{Hilb}^n(\mathbb{C}^2)$

This idea works more generally (Bialynicki-Birula, really just morse theory).

Computing $d(\alpha)$ is not hard, it turns out to be $d(\alpha) = n + \ell(\alpha)$ ← Length of partition

So $$\sum_{n=0}^\infty [\mathrm{Hilb}^n(\mathbb{C}^2)] t^n = \sum_{n=0}^\infty \sum_{\alpha \vdash n} \mathbb{L}^{n+\ell(\alpha)} t^n = \sum_{n=0}^\infty \sum_{\alpha \vdash n} \mathbb{L}^{\ell(\alpha)} (t\mathbb{L})^n$$

Quick digression on counting partitions. For $\alpha \vdash n$, there are several ways to encode the data of α

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\ell(\alpha)}) = (b_1(\alpha), b_2(\alpha), b_3(\alpha), \ldots\ldots) \qquad b_k(\alpha) = \#\text{ of parts of size } k$$

$$n = |\alpha| = \sum_{i=1}^{\ell(\alpha)} \alpha_i = \sum_{k=1}^{\infty} k\, b_k(\alpha) \qquad \ell(\alpha) = \sum_{k=1}^{\infty} b_k(\alpha)$$

The formula $\displaystyle\sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} (1-q^k)^{-1}$ works because

$$= \underbrace{(1+q+q^2+\cdots)}_{\text{take } b_1 \text{ term}}\underbrace{(1+q^2+q^4+\cdots)}_{\text{take } b_2 \text{ term}}\underbrace{(1+q^3+q^6+\cdots)}_{\text{take } b_3 \text{ term}}\cdots$$

This then easily generalizes

$$\sum_{n=0}^{\infty} \sum_{\alpha \vdash n} x^{\ell(\alpha)} q^n = \prod_{k=1}^{\infty} (1-xq^k)^{-1}$$

$$= (1+xq+x^2q^2+\cdots)(1+xq^2+x^2q^4+\cdots)(1+xq^3+x^2q^6+\cdots)\cdots$$

So
$$\sum_{k=0}^{\infty} [\mathrm{Hilb}^n(\mathbb{C}^2)]t^n = \sum_{n=0}^{\infty} \sum_{\alpha \vdash n} \mathbb{L}^{\ell(\alpha)} (t\mathbb{L})^n$$

$$= \prod_{m=1}^{\infty} (1-\mathbb{L}^{m+1}t^m)^{-1}$$

$$\Rightarrow \left(\sum_{k=0}^{\infty} [\mathrm{Hilb}^n_0(\mathbb{C}^2)]t^n\right)^{\mathbb{L}^2} = \prod_{m=1}^{\infty} (1-\mathbb{L}^{m+1}t^m)^{-1} \overset{\text{totaro}}{=} \prod_{m=1}^{\infty} (1-\mathbb{L}^{m-1}t^m)^{-\mathbb{L}^2}$$

$$\sum_{k=0}^{\infty} [\mathrm{Hilb}^n_0(\mathbb{C}^2)]t^n = \prod_{m=1}^{\infty} (1-\mathbb{L}^{m-1}t^m)^{-1}$$

$$\Rightarrow \left(\sum_{k=0} [\mathrm{Hilb}^n_0(\mathbb{C}^2)]t^n\right)^{[S]} = \prod_{m=1}^{\infty} (1-\mathbb{L}^{m-1}t^m)^{-[S]}$$

$$\sum_{k=0}^{\infty} [\mathrm{Hilb}^n(S)]t^n = \prod_{m=1}^{\infty} (1-\mathbb{L}^{m-1}t^m)^{-[S]}$$

$$\sum_{n=0}^{\infty} [\mathrm{Hilb}^n(S)]\, t^n = \prod_{m=1}^{\infty} (1-\mathbb{L}^{m-1} t^m)^{-[S]}$$

Recall if S is a smooth proj surface, then $\mathrm{Hilb}^n(S)$ is a smooth projective $2n$-fold so weight polynomial is Poincare poly. $b_i = \dim H^i(S)$ $\quad b_0 = b_4 = 1 \quad b_1 = b_3$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{4n} \dim H^k(\mathrm{Hilb}^n(S))\, s^k t^n = \prod_{m=1}^{\infty} (1 - s^{2m-2} t^m)^{-b_0 + b_1(-s) - b_2(-s)^2 + b_3(-s)^3 - b_4(-s)^4}$$

$$= \prod_{m=1}^{\infty} \frac{(1+s^{-1}(s^2t)^m)^{b_1}(1+s(s^2t)^m)^{b_1}}{(1-s^{-2}(s^2t)^m)(1-(s^2t)^m)^{b_2}(1-s^2(s^2t)^m)}$$

example if $S = K3$ $\quad b_1 = b_3 = 0 \quad b_2 = 22$ and if we let $s^2 = p = \exp(2\pi i z)$, $s^2 t = q = \exp(2\pi i \tau)$

$$\sum_{n=0}^{\infty} W_s(\mathrm{Hilb}^n(K3))\, t^n = \prod_{m=1}^{\infty} \frac{1}{(1-pq^m)(1-q^m)^{22}(1-p^{-1}q^m)}$$ is a theta function / Jacobi modular form.

Lecture 38

Using these ideas to compute Z^{DT} for local curves. Recall why we care about "local curves". Gromov-Witten theory is complicated because a single curve $C \subset X$ (CY3) contributes to infinitely many invariants through multiple covers and collapsing maps. Similarly, DT theory is complicated because there are many subschemes whose underlying reduced curve is C (nilpotent thickenings and embedded points). The main theorems in the field (correspondences between DT/GW/GV invariants) mediate and relate these complications

The way these theorems are proved are by using symplectic/analytic geometry to reduce the general case to local curves. So an important part of the story is solving DT & GW for local curves. (long history here). The original computation is very difficult, but powerstructures and topological vertex make this computation (for DT) easy (although it requires an unresolved Behrend function conjecture).

Suppose C is a smooth genus g curve and $L_1 \oplus L_2 \to C$ are line bundles such that $L_1 \otimes L_2 \cong K_C$ then $X = \mathrm{tot}(L_1 \oplus L_2 \to C)$ is a CY3 and is a "local curve". X isn't toric (unless $C = \mathbb{P}^1$), but it does have a $T = \mathbb{C}^\times \times \mathbb{C}^\times$ action. We wish to compute

Euler char weighted by $\nu : I_n(X, d[C]) \to \mathbb{Z}$

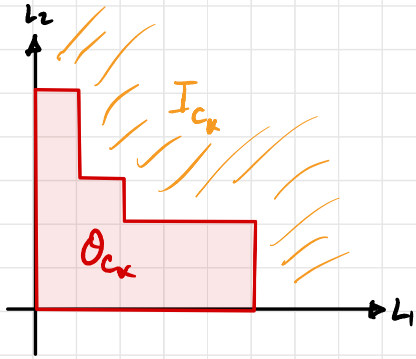
(we've used notation e_{vir})

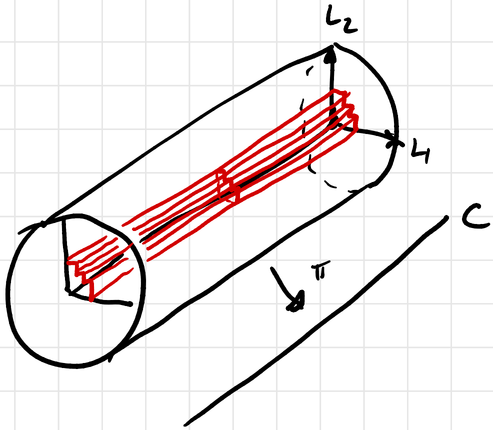
$$Z_d^{DT}(X) = \sum_n e(I_n(X, d[C]), \nu) \, (-q)^n$$

just write d instead of $d[C]$

$$= \sum_n e(I_n(X, d[C])^T, \nu) \, (-q)^n$$

What do T invariant subschemes look like? Pure dimension one subschemes which are invariant are in bijection with partitions $\alpha \vdash d$. Let C_α be the length d thickening of C which when restricted to a fiber of $\pi : X \to C$ is the monomial ideal determined by α:





Linear functions on the fibers of $X \xrightarrow{\pi} C$ are sections of $L_1^\vee \oplus L_2^\vee$; polynomial functions are sections of $\mathrm{Sym}^\bullet(L_1^\vee \oplus L_2^\vee)$ so $X = \mathrm{Spec}\left(\mathrm{Sym}^\bullet(L_1^\vee \oplus L_2^\vee)\right)$ and

$$\mathcal{O}_X \cong \mathrm{Sym}^\bullet(\pi^* L_1^\vee \oplus \pi^* L_2^\vee) \qquad \mathcal{O}_{C_\alpha} \cong \bigoplus_{i,j \in \alpha} \pi^* L_1^{-i} \otimes \pi^* L_2^{-j}$$

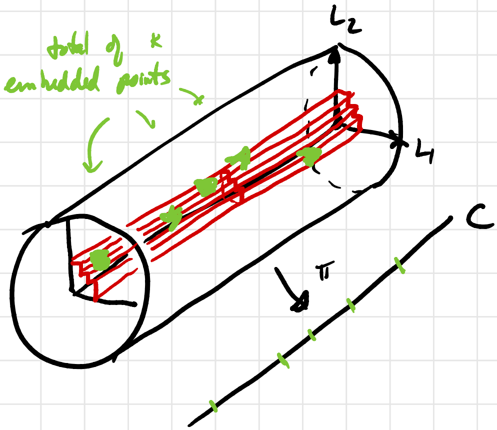
A general T invariant subscheme Z will be supported on C but can have (possibly very complicated) embedded points. However Z will have some C_α as the maximal pure dim 1 subscheme $C_\alpha \subset Z$.

$$0 \to P \to \mathcal{O}_Z \to \mathcal{O}_{C_\alpha} \to 0 \qquad \chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{C_\alpha}) + \operatorname{length} P$$

some 0-dim'l sheaf

Let $I_k(X, C_\alpha)^T \subset I_{\chi(\mathcal{O}_{C_\alpha})+k}(X, d[C])^T$ be the locus of T invariant subschemes $Z \subset X$ whose maximal pure subscheme is C_α and it has "k embedded points".

$I_k(X, C_\alpha)$ is still very complicated (not nec. locally monomial since we are only using a $(\mathbb{C}^\times)^2$).



Lecture 39

The motivic class of $I_k(X, C_\alpha)^T$ satisfies a power structure relation:

$$\sum_{k=0}^{\infty} [I_k(X, C_\alpha)^T] t^k = \left(\sum_{k=0}^{\infty} [I_k(\mathbb{C}^3, C_\alpha)_0^T] t^k \right)^{[C]}$$

where $I_k(\mathbb{C}^3, C_\alpha)_0^T$ parameterizes T invariant subschemes $Z \subset \mathbb{C}^3$ with $C_\alpha \subset Z$ maximal pure subscheme is given by the monomial ideal defined by α and supported on the z-axis and I_Z / I_{C_α} is length k and supported at 0.

Conjecture: the Behrend function is compatible with this structure.

$$Z_d^{DT}(X) = \sum_n{}' e\big(I_n(X, d[C])^T, \nu\big)(-q)^n$$

$$= \sum_{\alpha\vdash d} (-q)^{\chi(\mathcal{O}_{C_\alpha})} \sum_{k=0}^{\infty}{}' e\big(I_k(X, C_\alpha)^T, \nu\big)(-q)^k$$

conj

$$= \sum_{\alpha\vdash d}{}' (-q)^{\chi(\mathcal{O}_{C_\alpha})} \cdot \pm\Big(\sum_{k=0}^{\infty} e\big(I_k(\mathbb{C}^3, C_\alpha)_0^T, \nu\big)(-q)^k\Big)^{e(C)}$$

$$= \sum_{\alpha\vdash d} (-q)^{\chi(\mathcal{O}_{C_\alpha})} \cdot \pm\Big(\sum_{k=0}^{\infty} e\big(I_k(\mathbb{C}^3, C_\alpha)^{(\mathbb{C}^\times)^3}, \nu\big)(-q)^k\Big)^{2-2g}$$

Z locally monomial

The argument of MNOP then says that $\nu = (-1)^{\dim \mathrm{Ext}^1(I_Z, I_Z)} = (-1)^{\chi(\mathcal{O}_{C_\alpha}) + d(g-1) + k}$

for each $Z \subset X$ counted in the above. Then

$$Z_d^{DT}(X) = (-1)^{d(g-1)} \sum_{\alpha\vdash d}{}' q^{\chi(\mathcal{O}_C)} \Big(\sum_{k=0}^{\infty} \#\{I_k(\mathbb{C}^3, C_\alpha)^{(\mathbb{C}^\times)^3}\} q^k\Big)^{2-2g}$$

$$= (-1)^{d(g-1)} \sum_{\alpha\vdash d}{}' q^{\chi(\mathcal{O}_C)} V_{\phi\phi\alpha}^{2-2g}$$

Recall $\quad V_{\phi\phi\alpha}(q) = M(q) \prod_{i,j\in\alpha} \dfrac{1}{(1-q^{h_\alpha(i,j)})}$

line bundle of degree $-i\deg L_1 - j\deg L_2$

also $\quad \chi(\mathcal{O}_{C_\alpha}) = \chi(\pi_*\mathcal{O}_{C_\alpha}) = \chi\Big(\bigoplus_{i,j\in\alpha} L_1^{-i}\otimes L_2^{-j}\Big)$

$$= \sum_{i,j\in\alpha}{}' -i\deg L_1 - j\deg L_2 + 1 - g$$

Simplest case: since $L_1 \otimes L_2 = K_C$ we may choose some $K_C^{1/2}$ (a theta characteristic)

and let $L_1 = L_2 = K_C^{1/2}$. Then $\deg L_i = g-1$

so $\quad \chi(\mathcal{O}_{C_\alpha}) = (1-g) \sum_{i,j\in\alpha}{}' (i+j+1)$

Fun exercise: $\displaystyle\sum_{i,j\in\alpha}' (i+j+1) = \sum_{i,j\in\alpha}' h_\alpha(i,j)$ (Hint: how many different hooks is the i,j box contained in?)

so $\displaystyle \chi(\mathcal{O}_{C_\alpha}) = (1-g)\sum_{i,j\in\alpha}' h_\alpha(i,j)$

$L_i = K_c^{1/2}$

$$Z_d^{DT}(X) = (-1)^{(g-1)d} \sum_{\alpha\vdash d}' q^{\chi(\mathcal{O}_\alpha)}\, V_{\phi\phi\alpha}^{2-2g}$$

$$= (-1)^{(g-1)d} \sum_{\alpha\vdash d}' q^{(1-g)\sum_{i,j} h_\alpha(i,j)} \left(\prod_{i,j\in\alpha} \frac{1}{1-q^{h_\alpha(i,j)}} \right)^{2-2g} M(q)^{2-2g}$$

$$= M(q)^{2-2g} \sum_{\alpha\vdash d}' \left(\prod_{i,j\in\alpha} \frac{-q^{h_\alpha(i,j)}}{(1-q^{h_\alpha(i,j)})^2} \right)^{1-g}$$

$q^k = e^{ik\lambda}$

$$\frac{-q^k}{(1-q^k)^2} = \left(2\sin\frac{k\lambda}{2}\right)^{-2}$$

disconnected

$$Z_d^{GW}(X)' = \sum_{\alpha\vdash d}' \left(\prod_{i,j\in\alpha} 2\sin\frac{h_\alpha(i,j)\lambda}{2} \right)^{2g-2}$$

$2\sin\frac{h\lambda}{2} = h\lambda + \mathcal{O}(\lambda^3)$ so

$$Z_d^{GW}(X') = \sum_{\alpha\vdash d}' \left(\prod_{i,j\in\alpha} h_\alpha(i,j)\,\lambda \right)^{2g-2} (1+\mathcal{O}(\lambda^2))$$

$$\prod_{i,j\in\alpha} h_\alpha(i,j) = \frac{d!}{\dim R_\alpha}$$

irreducible S_d repr. indexed by α.

$$= \lambda^{d(2g-2)} \sum_{\alpha\vdash d}' \left(\frac{d!}{\dim R_\alpha} \right)^{2g-2} + \text{higher order } \lambda$$

A degree d unramified cover $C_h \xrightarrow{d:1} C_g$ has genus h satisfying $2h-2 = d(2g-2)$ and has the smallest genus of all degree covers.

So we get that the # of degree d, unramified covers of C_g is given by

$$\sum_{\alpha \vdash d} \left(\frac{d!}{\dim R_\alpha}\right)^{2g-2}$$

example $g=1$ this number is $p(d)$ which you did in homework

example $g=0$ formula says

$$\#\text{ of degree } d \text{ unramified covers of } \mathbb{P}^1 = \frac{1}{d!}\sum_{\alpha \vdash d}' (\dim R_\alpha)^2 = \frac{1}{d!}\cdot \frac{1}{|G|}\underbrace{\sum_{\substack{\text{irr} \\ \text{reps of } G}}' (\dim R)^2}_{\text{sum of squares of dim's of irr reps is } |G|.} \quad (G = S_d)$$

$$= \frac{1}{d!}$$ ← only the trivial cover which has an automorphism of S_d

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Lecture 40

Intro to PT theory. Recall that $Z' := \frac{Z^{DT}}{Z_0}$ ← "formally remove points". Is there a geometric theory whose partition function is Z'? Yes – Pandharipande–Thomas's theory of stable pairs.

Ideal sheaf I_Z can be equivalently viewed as $\mathcal{O}_X \overset{f}{\twoheadrightarrow} \mathcal{O}_Z$

moduli space of ideal sheaves can be viewed as moduli space of dimension 1 sheaves F equipped with a surjective morphism $f : \mathcal{O}_X \twoheadrightarrow F$

The bijective correspondence

$$\{f : \mathcal{O}_X \twoheadrightarrow F\} \longleftrightarrow \{I_Z\}$$
$$f \longmapsto \ker f$$
$$\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z \longleftarrow\!\shortmid I_Z$$

leads to an equivalence of moduli problems (note that $[\mathcal{O}_X \xrightarrow{f} \mathcal{O}_Z] \sim [I_Z]$ in $D^b(\mathrm{Coh}(X))$)

From the point of view of moduli of pairs $\{\mathcal{O}_X \xrightarrow{f} F\}$ $[\operatorname{supp} F] = \beta$ $\chi(F) = n$

the condition that f is surjective is a <u>stability condition</u>. The moduli stack of all pairs is not separated. To get a nice projective moduli space one needs an open set in the full stack which is separated and complete. There are other stability conditions besides f being surjective:

<u>Def'n</u> (PT) Let X be a 3fold. A stable-pair (F, f) is a sheaf F of dim 1 and a map $\mathcal{O}_X \xrightarrow{f} F$ such that
$f \in H^0(X, f)$

① F is pure ($\nexists$ non-zero $G \to F$ $\dim G = 0$)

② $\operatorname{coker} f$ is dim 0 (section doesn't vanish on components)

$$0 \to \ker f \to \mathcal{O}_X \xrightarrow{f} F \to \operatorname{coker} f \to 0$$

I_C C curve of pure dim 1

supp at points on C

"points" are still present, but now they can only occur on the curve.

<u>Thm</u> (LePotier, PT) Moduli space of stable pairs $PT_n(X,\beta)$ is projective. If X is CY3 it has a symmetric perfect obstr. theory with $\operatorname{Def}(\mathcal{O}_X \xrightarrow{f} F) = \operatorname{Ext}^1([\mathcal{O}_X \to F], [\mathcal{O}_X \to F])$ and $\operatorname{Ob}(\mathcal{O}_X \xrightarrow{f} F) = \operatorname{Ext}^2([\mathcal{O}_X \to F], [\mathcal{O}_X \to F])$.

$$\text{So } N^{PT}_{n,\beta}(X) = \int_{[PT_n(X,\beta)]^{vir}} 1 = e_{vir}(PT_n(X,\beta))$$

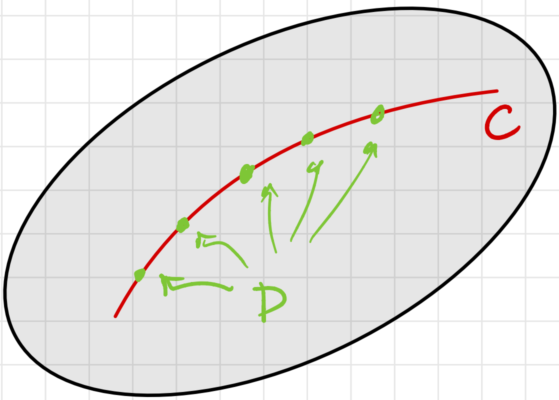
Let $Z^{PT}(X) = \sum_{n,\beta} N^{PT}_{n,\beta}(X)\, v^{\beta} (-q)^n$ $\qquad Z^{PT}_{\beta}(X) = \mathrm{coef}_{v^\beta} Z^{PT}(X)$

note that $Z^{PT}_0 = 1$ since the only non-empty moduli space with $\beta = 0$ is $PT_0(X, 0) = pt = \{[\mathcal{O}_X \to 0]\}$

Let $\mathcal{O}_X \xrightarrow{f} F$ be a stable pair supported on a smooth curve $C \subset X$.

Then F must be a line bundle on C with a section so $F = \mathcal{O}_C(D)$ where $D = \sum n_i p_i$ effective divisor $\qquad \mathcal{O}_X \xrightarrow{f} \mathcal{O}_C(D)$ in this case the data of the stable pair is a curve C with $N = \sum n_i$ points on it

$$n = \chi(F) = \chi(\mathcal{O}_C(D)) = 1 - g(C) + N$$



Simpler than subschemes.

<u>example</u> $\quad X = \mathrm{total}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{f'} & F' \\ & & \uparrow \cong \\ \mathcal{O}_X & \xrightarrow{f} & F \end{array} \qquad \text{so } f \sim rf \text{ if } r \neq 0.$$

$$PT_n(X, [\mathbb{P}^1]) = \{ \mathcal{O}_X \xrightarrow{f} \mathcal{O}_{\mathbb{P}^1}(n-1) \quad n \geq 1 \} = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(n-1))) = \mathbb{P}^{n-1}$$

$$Z^{PT}_{[\mathbb{P}^1]}(X) = \sum_{n \geq 1} e_{vir}(\mathbb{P}^{n-1})(-q)^n = \sum_{n \geq 1} (-1)^{n-1} n (-q)^n = -\sum_{n=1}^{\infty} n q^n = \frac{-q}{(1-q)^2}$$

Recall vertex computation: $\quad Z^{DT}_{[\mathbb{P}^1]}(X) = -M(q) \dfrac{q}{(1-q)^2}$

Theorem (Conj by PT, proved by Toda, Bridgeland).

$$Z^{DT}(X)' = \frac{Z^{DT}(X)}{Z_0^{DT}(X)} = Z^{PT}(X)$$

Change of stability $\leadsto$ wall crossing $\qquad \mathcal{O}_X \xrightarrow{f} F$ is a surjection in a perverse heart in $D^b(\mathrm{Coh}(X))$.

generalize previous example: local curve in degree 1 $\quad X = \mathrm{Tot}(L_1 \oplus L_2 \to C_g)$

$L_1 \otimes L_2 = K_C \quad H^0(C, L_i) = 0$ then only curve in class $[C]$ is $C \subset X$.

Thus $\quad PT_n(X, [C]) = \{ \mathcal{O}_X \xrightarrow{f} \mathcal{O}_C(D) \quad D$ effective divisor of deg $d = n+g-1 \}$

$$= \underset{\substack{\uparrow \\ \text{smooth of dim } n+g-1}}{\mathrm{Sym}^{n+g-1}(C)} \qquad n \geq 1-g$$

so $\quad N^{PT}_{n,\beta}(X) = (-1)^{n+g-1} e(\mathrm{Sym}^{n+g-1}(C))$

$$\sum_{k=0}^{\infty} e(\mathrm{Sym}^k(C)) t^k = (1-t)^{-e(C)} \quad \text{(power sr)}$$

$$Z^{PT}_{[C]}(X) = \sum_{n=1-g}^{\infty} (-1)^{n+g-1} e(\mathrm{Sym}^{n+g-1}(C)) (-q)^n \qquad d = n+g-1$$

$$= (-q)^{1-g} \sum_{d=0}^{\infty} e(\mathrm{Sym}^d(C))\, q^d$$

$$= (-q)^{1-g} (1-q)^{2g-2} = \left(\frac{-q}{(1-q)^2} \right)^{1-g}$$

$\leadsto \quad F^{GW}_{[C]} = Z^{GW}_{[C]} = \left(2 \sin \frac{\lambda}{2}\right)^{2g-2} \quad \Rightarrow \quad n_{[C],h}(X) = \begin{cases} 1 & h = g \\ 0 & h \neq g \end{cases}$