MATH 426 HOMEWORK 1

(1) Recall that if $A \subset X$ then we define $\overline{A}$ (the *closure* of A) to be the smallest closed set containing A. We define A° (the *interior* of A) to be the largest open set contained in A. We define A^c (the *complement* of A) to be $X - A$. Show that

(a)
$$A^\circ = \{a \in X \text{ such that } \exists\, U \text{ open}, a \in U \subset A\}$$
$$\overline{A} = \{x \in X \text{ such that } \forall\, U \text{ open with } x \in U, U \cap A \neq \emptyset\}.$$

(b) A is open if and only if $A = A^\circ$; A is closed if and only if $A = \overline{A}$.

(c) $(A^\circ)^c = \overline{A^c}$ and $\left(\overline{A}\right)^c = (A^c)^\circ$.

(2) A space X is called *irreducible* if $X = F \cup G$ wtih F and G closed implies that either $X = F$ or $X = G$. A *Zariski space* is a topological space such that every descending chain of closed sets $F_1 \supset F_2 \supset \dots$ is eventually constant. Show that every Zariski space can be expressed as a finite union

$$X = Y_1 \cup Y_2 \cup \cdots \cup Y_n$$

where Y_i is closed and irreducible (in the subspace topology) and $Y_i \not\subset Y_j$ for $i \neq j$. Show the decomposition is unique up to ordering.

(3) Recall that a space X is *connected* if the only subsets which are both open and closed are X and $\emptyset$. Recall that a space X is *path connected* if for all $p, q \in X$ there exists a map $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$.

(a) Prove that X is not connected if and only if there exists a surjective map fom X onto $\{0,1\}$, the space with 2 elements and the discrete topology.

(b) Let $X \subset \mathbb{R}^2$ be given by

$$X = \{0\} \times [-1,1] \cup \{(x, \sin\frac{1}{x}), x > 0\}$$

with the subspace topology. Show that X is connected but not path connected.

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