MATH 426 HOMEWORK 1 SOLUTIONS

(1) Recall that if $A \subset X$ then we define $\overline{A}$ (the *closure* of A) to be the smallest closed set containing A. We define A° (the *interior* of A) to be the largest open set contained in A. We define A^c (the *complement* of A) to be $X - A$. Show that

(a)
$$A^\circ = \{a \in X \text{ such that } \exists\, U \text{ open, } a \in U \subset A\}$$
$$\overline{A} = \{x \in X \text{ such that } \forall\, U \text{ open with } x \in U, U \cap A \neq \emptyset\}.$$

(b) A is open if and only if $A = A^\circ$; A is closed if and only if $A = \overline{A}$.

(c) $(A^\circ)^c = \overline{A^c}$ and $\left(\overline{A}\right)^c = (A^c)^\circ$.

Solution:

(a) We first show
$$A^\circ \subset \{a \in X \text{ such that } \exists\, U \text{ open, } a \in U \subset A\}.$$
For $a \in A^\circ$, we may take $U = A^\circ$ which is open by definition and so a satisfies the condition that there exists an open set U with $a \in U \subset A$.

We next show
$$A^\circ \supset \{a \in X \text{ such that } \exists\, U \text{ open, } a \in U \subset A\}.$$
Suppose there is an open set U such that $a \in U \subset A$. Since A° is by definition the largest open set contained entirely in A, and $U \subset A$, we conclude that $U \subset A^\circ$. Thus $a \in A^\circ$.

Now we show that
$$\overline{A} \subset \{x \in X \text{ such that } \forall\, U \text{ open with } x \in U, U \cap A \neq \emptyset\}.$$
Suppose $x \in \overline{A}$, and let U be an open set with $x \in U$. For contradiction we assume that $U \cap A = \emptyset$. Then $A \subset U^c$ and U^c is closed. Then by the definition of $\overline{A}$, $\overline{A} \subset U^c$. Then $x \in U$ and $x \in \overline{A} \subset U^c$, a contradiction.

Finally we show that
$$\overline{A} \supset \{x \in X \text{ such that } \forall\, U \text{ open with } x \in U, U \cap A \neq \emptyset\}.$$
We assume x satisfies the property that for all open U with $x \in U$, $U \cap A \neq \emptyset$. Suppose for contradiction that $x \notin \overline{A}$. Then $x \in \overline{A}^c$ which is open since $\overline{A}$ is closed by definition and so by the assumption on x we have that $\overline{A}^c \cap A \neq \emptyset$. This then implies that A is not a subset of $\overline{A}$, but by definition $A \subset \overline{A}$, a contradiction.

(b) Assume A is open, then A is automatically the largest open subset contained in A so $A^\circ = A$. Conversely assume $A = A^\circ$, then A is open by definition. Assume A is closed, since A is the smallest set that contains A and it is closed, it is $\overline{A}$. Conversely, assume that $A = \overline{A}$, then A is closed by definition.

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(c) We show $(A^\circ)^c = \overline{A^c}$:

$$\begin{aligned} x \in (A^\circ)^c &\Leftrightarrow x \notin A^\circ \quad \text{(then by the first part of (a))} \\ &\Leftrightarrow \forall U \text{ open with } x \in U, U \not\subset A \quad \text{(then by the second part of (a))} \\ &\Leftrightarrow x \in \overline{A^c} \end{aligned}$$

To show $(\overline{A})^c = (A^c)^\circ$, we take the compliment of both sides of $(A^\circ)^c = \overline{A^c}$ and then replace A with A^c.

(2) A space X is called *irreducible* if $X = F \cup G$ wtih F and G closed implies that either $X = F$ or $X = G$. A *Zariski space* is a topological space such that every descending chain of closed sets $F_1 \supset F_2 \supset \dots$ is eventually constant. Show that every Zariski space can be expressed as a finite union

$$X = Y_1 \cup Y_2 \cup \cdots \cup Y_n$$

where Y_i is closed and irreducible (in the subspace topology) and $Y_i \not\subset Y_j$ for $i \neq j$. Show the decomposition is unique up to ordering.

Solution: We wish to show that $X = F_1 \cup \cdots \cup F_N$ is a finite union of irreducible closed sets. Let $\mathcal{S}$ be the set of all closed subsets of X which *cannot* be written as a finite union of irreducible closed sets. Note that $\mathcal{S}$ is partially ordered by inclusion:

$$Z \leq Z' \Leftrightarrow Z \subset Z'.$$

Suppose for the sake of contradiction that $\mathcal{S}$ is non-empty. Then $\mathcal{S}$ has a least element Z in the partial ordering since if not, we could find a non-stabilizing sequence

$$X \subset Z_1 \supset Z_2 \supset \dots$$

Since Z cannot be written as a union of irreducibles, it is not irreducible itself and so

$$Z = F \cup G$$

where F and G are closed and $F, G \neq Z$. Then since Z is a least element, $F, G \notin \mathcal{S}$ and so they may be written as

$$F = F_1 \cup \cdots \cup F_n, \quad G = G_1 \cup \cdots \cup G_m$$

where F_i and G_j are irreducible. But then

$$Z = F_1 \cup \cdots \cup F_n \cup G_1 \cup \cdots \cup G_m$$

contradicting $Z \in \mathcal{S}$. Thus we conclude that $\mathcal{S}$ is empty and in particular we have

$$X = F_1 \cup \cdots \cup F_N$$

with F_i irreducible. If any $F_i \subset F_j$ we may remove F_i from the list and proceed until $F_i \not\subset F_j$ for all $i \neq j$.

To prove uniqueness we suppose that

$$\begin{aligned} X &= F_1 \cup \cdots \cup F_n \\ &= G_1 \cup \cdots \cup G_m \end{aligned}$$

with F_i and G_j irreducible. Then

$$\begin{aligned} G_i &= G_i \cap (F_1 \cup \cdots \cup F_n) \\ &= (G_i \cap F_1) \cup \cdots \cup (G_i \cap F_n) \end{aligned}$$

and by the irreducibility of G_i each $G_i \cap F_j = G_i$ or $G_i \cap F_j = \emptyset$.

(3) Recall that a space X is *connected* if the only subsets which are both open and closed are X and $\emptyset$. Recall that a space X is *path connected* if for all $p, q \in X$ there exists a map $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$.

(a) Prove that X is not connected if and only if there exists a surjective map fom X onto $\{0,1\}$, the space with 2 elements and the discrete topology.

(b) Let $X \subset \mathbb{R}^2$ be given by

$$X = \{0\} \times [-1,1] \cup \{(x, \sin\frac{1}{x}), x > 0\}$$

with the subspace topology. Show that X is connected but not path connected.

Solution: (a) Suppose there exists a surjective map

$$f : X \to \{0,1\}.$$

Then $f^{-1}(1)$ is not empty, nor is it X. But since $\{1\}$ is open and closed and f is continuous, $f^{-1}(1)$ is open and closed and is thus a component so X is not connected. Conversely, suppose X is not connected so that

$$X = C_0 \cup C_1$$

where $C_0 \cap C_1 = \emptyset$, $C_i \neq X$ or $\emptyset$, and C_i is both open and closed. Then define

$$f : X \to \{0,1\}$$

by $f(x) = i$ if $x \in C_i$. Since $f^{-1}(\{i\}) = C_i$ is open, f is continuous.

We note that (a) implies that the image of a connected space under a map is connected. Indeed, if X is connected and $f : X \to Y$ is a map with $f(X)$ disconnected, then there exists a surjective map $g : f(X) \to \{0,1\}$ and the composition $g \circ f : X \to \{0,1\}$ is surjective and continuous contradicting the connectedness of X.

We also note that an interval (closed or open) in $\mathbb{R}^1$ is connected since by ordinary calculus (e.g. the mean value theorem) there does not exist a continuous surjective map from an interval to $\{0,1\}$.

(b) Let $L = \{0\} \times [-1,1]$ and $C = \{(x, \sin\frac{1}{x}), \quad x > 0\}$ and so

$$X = L \cup C \subset \mathbb{R}^2$$

with the subspace topology. We note that $C \approx (0, \infty)$ via projection onto the x-axis and $L \approx [-1,1]$ via inclusion in the y-axis and so L and C are each connected spaces.

X is connected:

Suppose that X is not connected, then there exists a decomposition

$$X = V_0 \cup V_1$$

with $V_0 \cap V_1 = \emptyset$, V_i not X or $\emptyset$, and V_i both open and closed. Then

$$L = (L \cap V_0) \cup (L \cap V_1)$$

and since L is connected and each term in the above union is both open and closed, we have that either $L \cap V_0$ or $L \cap V_1$ is L. Without loss of generality, we may assume that $L = L \cap V_0$. By a similar argument with the connected space C we conclude that $C = C \cap V_1$ so that

$$V_0 = L \text{ and } V_1 = C$$

However, we get a contradiction since L is not open in X since every open set of X containing $(0,0) \in L$ contains $B_\epsilon(0,0) \cap X$ for some ϵ which necessarily contains points of C. Thus X is connected.

X is not path connected:

Suppose that X were path connected. Then there exists a path

$$\gamma : [0,1] \to X$$

such that $\gamma(0) = (0,0) \in L$ and $\gamma(1) = (\frac{1}{\pi}, 0) \in C$. Since L is closed,

$$\gamma^{-1}(L) \subset [0,1]$$

is closed and thus contains its least upper bound b. Then the restriction of γ to $[b,1]$ is a map (which we still call γ)

$$\gamma : [b,1] \to X$$

such that $\gamma(b) = (0,a) \in L$ and $\gamma((b,1]) \subset C$. Let $\pi : X \to [0,\infty)$ be the projection onto the x-axis. Since $[b,1]$ is connected, the image of $[b,1]$ under the composition $\pi \circ \gamma$ is connected and contains both 0 and $\frac{1}{\pi}$ and thus contains all $0 \leq x \leq \frac{1}{\pi}$. Thus $\gamma([b,1]) \subset X$ contains both $(0,a)$ and points $(x, \sin\frac{1}{x})$ with $0 < x \leq \frac{1}{\pi}$. Since $[b,1]$ is compact and $\mathbb{R}^2$ (and hence X) is Hausdorff, $\gamma([b,1])$ is closed. We reach a contradiction by showing that $\gamma([b,1])^c$ is not open. To that end pick $a' \neq a$ with $a' \in [-1,1]$ so that $(0,a') \in \gamma([b,1])^c$. Then any open set in X containing $(0,a')$ contains some ball $B_\epsilon((0,a')) \cap X$. But then this ball intersects $\gamma([b,1])$ since we can find some $0 < \delta < \epsilon$ such that $(\delta, \sin\frac{1}{\delta}) \in B_\epsilon((0,a'))$. This contradicts the openness of $\gamma([b,1])^c$.